Beyond the Beach: Pricing Competition in Two Dimensions

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March 10, 2025

Abstract

We develop a generalized framework for spatial price competition that extends Hotelling's classic model to a two-dimensional market with heterogeneous population density. Firms, offering an identical homogeneous good, operate in a continuous space, with neither their locations nor the distribution of consumers subject to predetermined constraints. The model accommodates arbitrary spatial configurations, ensuring broad applicability. Firms set prices to maximize profits and compete for spatial demand, which is characterized by "pseudo-Voronoi" regions where consumers are more likely to purchase from the closest firm. We study existence and uniqueness of the equilibrium, finding similar equilibrium characteristics as for the linear case. By extending Hotelling's model, our framework tries to enhance its empirical applicability.

1 Introduction

Spatial competition has long been a central topic in industrial organization and economic geography, with important implications for market structure, firm behavior, and urban planning (among the others Eaton and Lipsey (1975), Irmen and Thisse (1998),Eiselt (1989), Jensen (2006))

One of the earliest and most influential models in this field is Hotelling (1929) linear-city model, which introduced a simple but powerful framework to analyze how location and price competition shape market outcomes. A central takeaway from his work is the tendency of firms to converge toward the market's center, a phenomenon known as "minimal differentiation" or the "principle of median location". Hotelling's model is fundamental for analyzing market dynamics in various contexts, including retail location strategies, political competition, and online platforms. However, this result is highly sensitive to the assumptions underlying the model, such as uniform consumer distribution, linear city structures, and deterministic consumer behavior. Furthermore, some models *à la* Hotelling while using more realistic specifications are limited to the case of the two firms, such as Tabuchi (1994), Larralde et al. (2009) and Houba et al. (2023).

In his original work, Hotelling (1929) argued that an equilibrium would emerge in which both firms position themselves arbitrarily close together at the center of the market. He illustrated this concept through a brand positioning game, where two firms producing and selling cider located their products along a tart-sweet continuum (Drezner and Eiselt (2024)). The problem of the model occurs, though, exactly when facilities are located close to each other. d'Aspremont et al. (1979) showed that the equilibrium actually does not exist. A (Nash) equilibrium is defined as a situation in which neither of the competitors (mostly duopolies in the early work) can improve its objective by unilaterally changing its course of action. A common illustration of this concept is the competition between ice cream vendors along a beach, modeled as a straight-line segment. Assuming that each tourist purchases from the nearest vendor, each competitor strategically selects a location to maximize their customer base. In a Nash equilibrium, no vendor can improve their profit by unilaterally relocating. As Barro (2024) notes, a key feature of the original model is that each store's pricing strategy is shaped by competition from neighboring stores. From the consumer's perspective, the most direct substitutes for any given store are its adjacent competitors.

An underdeveloped branch in the spatial competition literature is to generalize Hotelling's ideas to accommodate arbitrary market geometries, heterogeneous consumer distributions, and probabilistic choice mechanisms within a unified theoretical structure. Despite the elegance of the original linear model, real-world markets often depart from their one-dimensional assumptions, or they are two-dimensional and feature heterogeneous consumer densities. Over the past decades, numerous extensions and modifications of Hotelling's model have been proposed to address its limitations and provide a more general framework for spatial competition. The most well-known modification is the Salop (1979) circular-city model, which replaces the finite-line structure with a circle to eliminate boundary effects and better capture competition in markets with free entry. Other studies have explored alternative spatial structures, such as hexagonal market areas in central place theory and triangular configurations. Many of these contributions focus on adapting Hotelling's framework to different spatial formulations, shifting away from strict linearity toward more flexible spatial forms. This new wave of research has moved beyond the traditional assumption of Hotelling model started incorporating rich geography into firm competition and location choice models. This shift has been particularly evident in the fields of urban economics, international trade, and regional science, where scholars have emphasized the role of transportation networks, spatial frictions, and heterogeneous consumer densities (or positioning) in shaping market outcomes (i.e Allen and Arkolakis (2014), Redding and Rossi-Hansberg (2017), Alderighi and Piga (2008)). Most of the models that build on Hotelling's framework still rely on predefined discrete spatial structures (e.g., line, circle, hexagon) rather than allowing the market structure itself to be an endogenous outcome of firm competition. Hotelling's result does not hold under realistic conditions but was able to re-establish it given uncertainty and some behavioral assumptions while Bester et al. (1996) pointed out that Hotelling's model does require some coordination between the competitors, as they have to agree who locates to the left and to the right of each other. Moreover, d'Aspremont et al. (1979) criticized Hotelling's finding and showed that when players compete on price as well as location, they tend to create distance from one another, otherwise price competition would drop their profit to zero.

As underlined Michler and Gramig (2021)), since Hakimi (1983), most location games in real space have defined the problem in terms of medianoid and centroid polygon shaped areas to be maximized. These extensions offer significant technical advantages, particularly by addressing issues related to the extreme endpoints of a finite linear space, as seen in Salop (1979) model which replaces the finite line with a circular market structure, effectively eliminating "boundary problems" or distortions near the edges. In Salop's type model, however, people are only allowed to travel along the circle when shopping, while pass-through travel seems more plausible than traveling along the circle, because of the lesser distance involved in indirect point-to-point travel. On this view, a good way to consider the extension to a two-dimensional Euclidean space. In line with this perspective, Tsai and Lai (2005) introduced a triangular market structure as an initial step toward broader generalization and recent innovations(Peng et al. (2020)). In models proposed in central place theory, hexagonal arrangements of market areas around a central place represent the optimal spatial organization for a single good, under the assumption of uniform densities on an unbounded plain with equal access in all directions. DeSerpa (1986) argued that a general equilibrium could be remained unaffected by the market's shape since the specific areas firms consider their domain, provided that a regular polygon is used in a two-dimensional space similar to the case proposed by Capozza and Van Order (1978).

These geometrical adaptations of Hotelling game in 2D, however, don't account directly for the endogeneity in the sharpness of the market areas. Taking up what Drezner and Drezner (2017) and Drezner and Eiselt (2024), since customers are attracted to the closest facility, the market share captured by each facility is proportional to the area attracted to the closest facility. This is similar to the Voronoi diagram concept.

Voronoi tessellation is an underexplored instrument to apply in order to endogenize the market area. Following the definition by (Ito (2015)), for a set P of points in the n-dimensional Euclidean space, the Voronoi tessellation is the partition $V(P)$ of the space such that each point in P has a region which is closer to that point than to any other points in P.

As shown by Eiselt (1989) and, more recently, by Burkey et al. (2011), Voronoi diagrams are a tool that can be used both for the localization/spatial evolution of cities (i.e, the recent contribution of Lanzara and Santacesaria (2023)) and in the analysis of competition through game theory. In the literature review proposed by Abudaldah et al. (2015), they underlined as Okabe and Suzuki (1987) showed with Voronoi-diagrams that, for very large numbers of firms, the configuration has stable equilibria in the inner area of square regions of a model *à la* Hotelling. Tabuchi (1994), Irmen and Thisse (1998) and Veendorp and Majeed (1995) all show that, under specific conditions, firms in a bounded two-dimensional space minimize differentiation on one dimension, and maximize it on the other. Drezner and Drezner (2017) work (applied to alLeader-follower Model of facility location) doesn't represent the first attempt in introducing Voronoi diagram in the study of spatial competition (with some game theory application). Eiselt (1989) proposed that Voronoi diagrams could be applied on managerial problems as construction, query and optimization. Voronoi areas provide a valuable framework for analyzing market structures and defining relevant markets also in empirical modeling. Brandt et al. (2014) compares firm entry models in Sweden's wholesale sector, examining the differences in results when market boundaries are defined using administrative regions versus Voronoi polygons. Auerbach and Dix (2017) employs tessellation based on Lloyd's algorithm (Voronoi relaxation) to assess spatial inefficiency, applying this method to various case studies, including gas stations, supermarkets, fire stations, and hospitals across the United States. Kvasnička et al. (2018) investigates the impact of station density on gasoline prices in the Czech Republic's retail fuel market partitioning the area into Voronoi polygons around the gas stations of the same brand.

Hotelling's games, when applied to spatial competition among firms, allow for the formation of Voronoi cells. This is because each firm's payoff corresponds to the length of the segment where consumers are closer to it than to any competitor (Avin et al. (2022)). Moreover, an interesting topic related to some issues that this paper tries to face is uncertainty in client behaviors (Meagher and Zauner (2005)). We will reprise this argument later, but is useful because of agent-based models of spatial competition have also been employed on Hotelling's game, such as in van Leeuwen and Lijesen (2016) (in a case of *locate-then-price* setting) and, on a microsimulation approach, Ottino-Loffler et al. (2017).

Voronoi games, introduced by Ahn et al. (2004) as a two-player symmetric game on a segment, where the players locate facilities alternatingly and are not able to locate a facility on an already occupied location. Moreover, further case analyzed by Feldmann et al. (2009), can be seen as a discrete, plane-based adaptation of Hotelling's spatial competition model. These games resemble classical Hotelling games, where multiple vendors compete within a continuous metric space. Each vendor, offering goods for sale, must simultaneously choose a location for their facility. The goal is to maximize the share of the market, defined as the region of points that are closer to the vendor than to any competitor—this region is known as the vendor's Voronoi cell. Núñez and Scarsini (2016)and Núñez and Scarsini (2017) proposed an interesting framework of a location game *à la* Hotelling, where, despite is assumed that competition among retailers is only in terms of location and not price [1] where Retailers can choose one of finitely many locations in this space (differently by classical Hotelling's game) and prove symmetric mixed equilibria which exist for any number of retailers and that the distribution of retailers tends to agree with the distribution of the consumers when the number of competitors is large enough.

As highlighted by Roncoroni and Medo (2016) the mutual position of the firms determines the equilibrium profit of each of them when the number of firms rises (in Hotelling's game with linear cost function and periodic boundary conditions). The situation is more complex than in the case of two firms where their distance is the only variable. Even when the firms are placed at random, the equilibrium price of a particular firm is decided by the distances of several firms that surround it. In particular, the firms that lie in the adjacent regions of the Voronoi tessellation of the plane are the ones with which the studied firm has direct contact and competes for customers. The equilibrium prices of those firms are further decided by their neighboring competing firms, and so forth, and it is thus easy to see that the situation is not analytically approachable without making a further simplifying assumption. Since the strongest competition is with the closest neighboring firm (equilibrium profit increases with distance between the firms), our simplifying assumption is that the equilibrium profit is decided solely by the distance of the closest firm.

Avin et al. (2022) built an n-player fault-prone Hotelling game (account for the possibility of faults in their behavior). In such a game, n-servers choose a location along the $[0, 1]$ segment, and as their payoff they get the expected size of their Voronoi cell in the presence of random line faults (or a disconnection at a point on the line through which clients cannot pass).

As evidenced by Boots and South (1997), the multiplicatively weighted Voronoi area represents a special case of standard tessellation. The diagram considers both locational and non-locational attributes of facilities (represented as an aggregate measure of attractiveness) and assumes that customers select stores on the basis of a trade-off between distance and attractiveness. The boundary of the market areas represent, so, an issues both for firm positioning (as in the original Hotelling model) and consumer behavior since consumers, in choosing to seek the lower price, finding themselves on the boundary of the Voronoi area also react strongly to a marginal shift in the price, distance or weight of these two variables.

The boundary areas thus defined, therefore, are a serious impediment to the designation of a stable equilibrium in the model, since generally clients shop at the closest vendor, but with some probability on their own actions. Due to unquantifiable factors of personal taste, they skip a seller and travel further to the next

[1] A relevant part of the literature about Hotteling's games are two-stage models, where competition first happens on location and subsequently on price, or *locate-then-price game* (Heywood et al. (2021).

one. These forms of consumer "irrationality" has been studied by considering several aspects. One of these is the fuzzy component in case of demand, imprecise preferences, fuzzy utility for consumer and locations (i.e Ponsard (1995),Uno et al. (2012), Nasiri et al. (2018)). The conditions for the existence of spatial general equilibrium are less restrictive in a fuzzy economy compared to a traditional economy. The incorporation of fuzzy economic calculations allows for a more general theoretical framework and helps address complex issues, particularly the challenge of equilibrium production under increasing returns. However, despite this greater flexibility, equilibrium conditions remain sufficiently stringent, meaning that, much like in conventional economic spaces, a fuzzy economic space is often in a state of disequilibrium.

Roncoroni and Medo (2016) addresses this problem using a Hotelling model on a plane with periodic boundary conditions (PBC), which are particularly advantageous for numerical simulations as they help mitigate finite-size effects. The study also generalizes transportation costs, modeling them as a power function of distance, where linear and quadratic costs emerge as special cases. firms, a key result is that the total equilibrium profit of all firms decreases as $1/\sqrt{m}$ where m represents the number of competing firms. Numerical analysis of generalized transportation costs—growing as a power of distance—further reveals that the equilibrium profit per firm follows the form $1/m^B$ with the exponent B generally increasing alongside the exponent of the transportation cost function. More broadly, the study highlights the relevance of modeling spatial competition with linear transportation costs, which are arguably more natural and widespread than quadratic ones. Periodic pricing patterns emerge, where firms gradually increase prices over multiple iterations until it becomes profitable for the least favorably positioned firm (i.e., the one closest to the unit square boundary) to drastically lower its price and capture the entire customer base. While the findings suggest that transportation costs are more likely to resemble quadratic rather than linear functions, the approach does not account for consumer choice probabilities. Despite its valuable insights, the issue of boundary effects remains unresolved for our aims through the use of PBCs.

Both of these approaches—applied to Voronoi areas, as seen in Fu et al. (2010) for periodic boundary conditions (PBCs) and Fan et al. (2016) for fuzzy models—lack of direct economic applications that are on that are of interest to us. For this reason, the logit model (Anderson et al. (1992)) emerges as a practical alternative. This framework has already been explored in spatial competition and urbanization models, as seen in Wrede (2015) and Zhang and Zhao (2012).

In particular, Larralde et al. (2009) propose a model in which optimal firm locations depend on consumer heterogeneity. This heterogeneity introduces additional choice factors, modeled as random "noise" in utility functions, with its magnitude captured by the parameter μ. When heterogeneity is sufficiently high, the market-stealing effect dominates: consumers perceive the two stores as sufficiently distinct, reducing direct price competition even when they are geographically close. Conversely, when heterogeneity is low, the market-power effect prevails, making firms prefer spatial separation to soften price competition.

In our case, integrating the logit framework helps address the boundary problem. To our knowledge, weighted Voronoi diagrams have not yet been applied in conjunction with this type of probability model, which is why we introduce the concept of "pseudo-Voronoi" diagrams.

In this paper, we present a unifying 2D spatial competition framework that preserves Hotelling's core ideas but relaxes some of its assumptions. We adopt a continuous geography $S \subset \mathbb{R}^2$, let consumer density $\rho(s)$ vary over S, and let firms choose their prices but not their locations.

This paper seeks to bridge these gaps by proposing a unifying spatial competition framework that generalizes Hotelling's core insights while relaxing several restrictive assumptions. We develop a two-dimensional competitive model where firms compete in a continuous geographic space, and consumer density varies smoothly across the market. We argue that a generalization of Voronoi regions, the multiplicatively Voronoi regions, better describe the rational behavior of consumers. However that this kind of spatial market areas makes the model equilibrium unstable.

Crucially, market areas are endogenously determined through multiplicative-weighted Voronoi tessellations, which naturally generalize Hotelling's discrete partitioning in one dimension. We provide conditions for the existence of a pure-strategy price equilibrium for any set of firm locations, and show that under additional conditions, the price equilibrium is unique. The location problem can then be studied in a dynamic sense, with firms entering sequentially. By integrating rich geography, endogenous market structures, and probabilistic demand, this paper presents a more general framework for spatial competition, extending traditional Hotelling-inspired models.

The paper is organized as follows. Section 2 sets up the model. Section 2.1 states and proves the existence/uniqueness results (adapted from logit-based game-theoretic arguments). Section 3 examines the monopoly case, while Section 4 discusses comparative statics and possible dynamic extensions. Section 5 shows how Hotteling's and Salop's original models emerge as particular case of the presented model. Section 7 shows the numerical simulations.

2 The model

Consider a compact, convex region $S \subset \mathbb{R}^2$ describing a geographic market. Consumers are located in the space and are immobile. Let $\rho : S \to \mathbb{R}_+$ denote a continuous, bounded consumer density with total mass

$$M \;=\; \int_S \rho(s)\, ds.$$

We assume $M > 0$ is finite.

In the market, there is an exogenously given number N of firms selling the same homogeneous good, and they are indexed by $\mathcal{N} = \{1, \ldots, N\}$, each located at $x_i \in S$. Firms compete among themselves, setting their prices $p_i \in [c, \overline{p}]$, where $c > 0$ is the equal marginal cost among firms, and $\overline{p}$ is a large enough upper bound such that, beyond it, demand is zero or negligible.

Consumers have constant elasticity demand $q(s) = A[P_i^*(s)]^{-\eta}$, with $\eta > 1$ and $A > 0$, where $P_i^*(s)$ is the effective price faced by consumer i buying from a firm located at s.

A function $d : S \times S \to [0, \overline{d}]$ represents the fraction of "value" lost in shipping a unit from location i (where firm x_i is located) to consumer location s. We assume d is continuous in both arguments and convex in the firm's location x. The *effective price* paid by a consumer located in s buying from the firm x_i depends on both the firm's price and its distance from the consumer to the firm's location i:

$$P_i^*(s) = p_i\,[1 + d(s, x_i)], \tag{1}$$

where $p_i > 0$ is the price set by firm i.

Previous research following Hotelling suggested that consumers buy from the closest firm, essentially purchasing from the firm whose Voronoi region they fall into. A Voronoi region for a point is defined as the set of locations which are nearest to that point than to any other point. Hence, all consumers will buy only from the firm they are nearest to or, in other terms, from the firm whose Voronoi area they fall into. Taking a step forward in economic intuition, we argue that, following directly from equation 1, consumers choose instead the firm that sells the homogeneous good at the lowest effective price. This specification accounts for both price and distance factors. Therefore, we suggest that multiplicative weighted Voronoi regions better model consumer rational behavior. This region includes all consumers who will face lower effective prices when buying from firm i than from any other firm:

$$V_i(\mathbf{p}) = \bigl\{\, s \in S : \; p_i(1 + d(s, x_i)) \;\leq\; p_j(1 + d(s, x_j)),\, \forall j \bigr\}. \tag{2}$$

The economic mechanism has quite an intuitive representation. If prices are constant among firms, each firm's market area corresponds to the simple Voronoi tessellation. However, if prices are free to change, firms compete with each other to increase their market areas until their profits are maximized. As can be seen in the case on the right of figure 1, if firm x_3 decreases its price, its market area increases, reducing those of the other firms'. While this approach generalizes standard Voronoi diagrams by including price in the distance weighting, it creates a rather strict formulation particularly at regional borders. Consumers living at these boundaries would have an absolute preference toward a firm that might be only marginally more economical. Furthermore, this formulation presents challenges for equilibrium analysis, as even small price changes by one firm can abruptly shift market boundaries, potentially violating Glicksberg's continuity assumption. To address these concerns, we introduce the concept of Pseudo Voronoi regions that assign probabilities to consumer buying decisions:

$$\mathrm{Prob}\bigl(i \mid s, \mathbf{p}\bigr) \;=\; \frac{\exp\bigl(-\beta\, P_i^*(s)\bigr)}{\sum_{j=1}^{N} \exp\bigl(-\beta\, P_j^*(s)\bigr)}, \tag{3}$$

where $\beta > 0$ is the logit parameter. A similar approach has been used by Larralde et al. (2009). A notable property is that, as $\beta \to \infty$, the consumer at s chooses the firm i with the lowest $P_i^*(s)$ with probability 1, thereby recovering the "lowest delivered price wins" scenario.

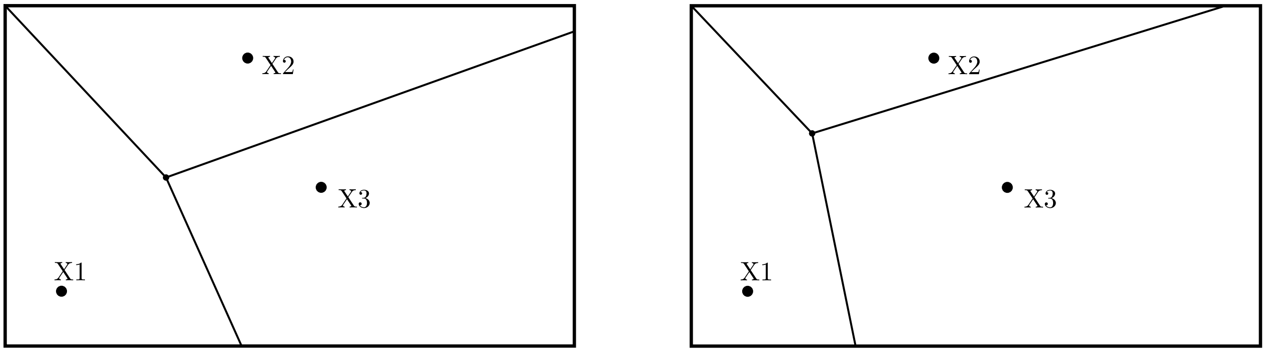

Figure 1: Multiplicative Voronoi market areas. The left panel shows the case when all firms have the same price. The right panel shows the case when firm X3 lowers its price.

We refer to equation 3 as "pseudo Voronoi" because, in locations where the effective price is considerably lower than that at any other location, the function closely resembles the multiplicative weighted Voronoi formulation. In other words, consumers residing in areas where the effective prices of purchasing from a specific firm are significantly lower than those of other firms tend to strongly favor buying from that firm. However, at price boundaries—specifically in locations where the price differences among two or more firms are less pronounced—consumer preferences are weaker.

After having defined the consumer choice function we can define the *spatial demand* for firm i. Given $\mathbf{p} = (p_1, \ldots, p_N)$, the total demand for firm i is

$$Q_i(\mathbf{p}) \;=\; \int_S \rho(s) \left(A \, [P_i^*(s)]^{-\eta} \right) \mathrm{Prob}\big(i \mid s, \mathbf{p}\big) \, ds.$$

We define profit for firm i as

$$\Pi_i(\mathbf{p}) \;=\; (p_i - c) \, Q_i(\mathbf{p}).$$

The firm's objective is to maximize its profits by choosing an optimal price p_i. This leads to the profit maximization problem:

$$\max_{p_i} \Pi_i(\mathbf{p}) = \max_{p_i} (p_i - c), Q_i(\mathbf{p})$$

2.1 Price Equilibrium: Existence and Uniqueness

We analyze a simultaneous-move price game, taking $\{x_i\}_{i=1}^N$ as given. Each firm chooses $p_i \in [c, \overline{p}]$ to maximize $\Pi_i(\mathbf{p})$. An equilibrium is a vector $\mathbf{p}^*$ such that no firm can profitably deviate.

2.1.1 Existence Theorem and Proof

Theorem 1 (Existence of Price Equilibrium)**.** *Under the assumptions above (continuous ρ, $\eta > 1$, $\beta > 0$, and compact strategy sets $[c, \overline{p}]$), and in particular a logit consumer choice function, there exists at least one pure-strategy price equilibrium.*

Proof. (*Following Glicksberg theorem (Glicksberg, 1952)*the complete proof is given in Appendix A
Step 1: Compact, Convex Strategy Sets. Each firm i chooses p_i from the interval $[c, \overline{p}]$, which is compact and convex.
Step 2: Continuity of Profit Functions. $\Pi_i(p_i, \mathbf{p}_{-i})$ is a continuous function of all prices:

- $P_i^*(s) = p_i(1 + d(s, x_i))$ depends continuously on p_i.

- $q(s) = A[P_i^*(s)]^{-\eta}$ is continuous in p_i.
- $\text{Prob}(i \mid s, \mathbf{p})$ is continuous in $\mathbf{p}$ (the logit functional form).
- The integral $\int_S \rho(s)\, q(s)\, \text{Prob}(i|s,\mathbf{p})ds$ is then continuous by dominated convergence (boundedness of ρ, d, and p_i).

Step 3: Quasi-Concavity in Own Price. To apply a fixed-point argument, we also need each firm's best-response to be a weakly convex-valued correspondence. One possibility is to check that $\Pi_i(\mathbf{p})$ is *strictly concave* in p_i (or $\ln \Pi_i$ is concave). Indeed, the second derivative in p_i (holding $\mathbf{p}_{-i}$ fixed) is negative under $\eta > 1$ and mild regularity on the logit function.
Step 4: Glicksberg's Fixed Point Theorem. Having a continuous game with compact, convex strategy sets and quasiconcavity of payoffs in own strategies (or a single-valued continuous best-response), Glicksberg's theorem guarantees at least one pure-strategy Nash equilibrium. □

2.1.2 Uniqueness Theorem and Proof

Theorem 2 (Uniqueness of Price Equilibrium)**.** *Suppose, in addition to the conditions of Theorem 1, that the elasticity of demand* η *is sufficiently high relative to the competitive overlap between firms. Formally, if*

$$\eta > 1 + \beta \max_i \left\{ \frac{\sum_{j\neq i} \omega_{ij}}{\omega_{ii}} \right\},$$

where $\omega_{ij} = \int_S \rho(s) Prob(i|s,\mathbf{p}) Prob(j|s,\mathbf{p})ds$ *represents the competitive overlap between firms* i *and* j*, and* $\omega_{ii} = \int_S \rho(s) Prob(i|s,\mathbf{p})(1 - Prob(i|s,\mathbf{p}))ds$ *represents firm* i*'s own-market sensitivity, then the price equilibrium is* unique*.*

Proof. We provide a sketch of the proof here; the complete proof is given in Appendix B.

Let $F_i(\mathbf{p}) = \frac{\partial \Pi_i(\mathbf{p})}{\partial p_i}$ be firm i's first-order condition. We analyze the Jacobian matrix $\mathbf{J}(\mathbf{p}) = \left[\frac{\partial F_i(\mathbf{p})}{\partial p_j}\right]_{i,j\in\mathcal{N}}$ and demonstrate that it is a P-matrix at any candidate equilibrium. Specifically, we show that $\mathbf{J}$ is strictly diagonally dominant with negative diagonal elements:

$$\left|\frac{\partial F_i}{\partial p_i}\right| > \sum_{j\neq i} \left|\frac{\partial F_i}{\partial p_j}\right| \quad \forall i \in \mathcal{N}$$

By examining the relationship between demand elasticity, competitive overlap, and the sensitivity parameter, we establish that the condition $\eta > 1 + \beta \max_i \left\{ \frac{\sum_{j\neq i} \omega_{ij}}{\omega_{ii}} \right\}$ ensures diagonal dominance. When firms are sufficiently differentiated in space, the uniqueness of equilibrium follows from standard fixed-point theorems. □

3 Special cases and analytical solutions

Finding analytical solutions for the model is very difficult, perhaps impossible. In the next pages we show some analytical results for special cases of the model.

Monopoly

We first analyze the monopoly case ($N = 1$), where a single firm chooses both its location $x_1 \in S$ and price $p_1 \geq c$ to maximize profits. This benchmark clarifies how spatial costs and demand elasticity interact in pricing and location decisions.

Optimal Pricing for a Fixed Location

Consider a monopolist at location x_1. Its profit function is:

$$\Pi(p_1) = (p_1 - c) \int_S \rho(s) A \big[p_1(1 + d(s, x_1))\big]^{-\eta} ds,$$

where delivered prices $p_1(1 + d(s, x_1))$ dampen demand across space. The key tradeoff is between extracting surplus (higher p_1) and losing distant customers (lower p_1).

Proposition 1 (Monopoly Price Formula)**.** *For any location $x_1 \in S$, the profit-maximizing price p_1^* satisfies:*

$$\underbrace{\int_S \rho(s)[p_1(1+d)]^{-\eta} ds}_{Demand} = \eta(p_1 - c) \underbrace{\int_S \rho(s)[p_1(1+d)]^{-\eta-1}(1+d) ds}_{Marginal\ revenue\ loss}.$$

For population density $\rho(s)$:

- *If $\rho(s)$ is uniform ($\rho(s) = \rho$ for all $s \in S$) and transportation costs are spatially uniform ($d(s, x_1) \approx$ constant) or negligible, this simplifies to:*

$$p_1^* = \frac{\eta}{\eta - 1} c \quad \textit{(constant elasticity markup)}.$$

- *If $\rho(s)$ is non-uniform, the optimal price depends on the spatial distribution of customers and generally cannot be expressed as a simple markup.*

Proof. Differentiate $\Pi(p_1)$ with respect to p_1:

$$\begin{aligned} \frac{\partial \Pi}{\partial p_1} &= \int_S \rho(s) A [p_1(1+d)]^{-\eta} ds \\ &\quad - \eta(p_1 - c) \int_S \rho(s) A [p_1(1+d)]^{-\eta-1}(1+d) ds. \end{aligned}$$

Setting $\partial\Pi/\partial p_1 = 0$ yields the general FOC.

For the uniform density case, $\rho(s) = \rho$ can be factored out of all integrals:

$$\rho \int_S [p_1(1+d)]^{-\eta} ds = \rho\eta(p_1 - c) \int_S [p_1(1+d)]^{-\eta-1}(1+d) ds$$

With ρ canceling out, the pricing decision becomes independent of the density level.

Under uniform transportation costs, factor out $(1 + d)$:

$$[p_1(1+d)]^{-\eta} = \eta(p_1 - c)[p_1(1+d)]^{-\eta-1}(1+d),$$

which collapses to $p_1^* = \frac{\eta}{\eta-1} c$.

For non-uniform density, $\rho(s)$ remains inside the integrals, and the optimal price depends on the specific spatial distribution. The second derivative $\partial^2\Pi/\partial p_1^2 < 0$ confirms optimality in all cases. □

Optimal Location Choice

The monopolist concurrently chooses where to locate. Intuitively, central locations minimize average transportation costs, maximizing accessible demand.

Proposition 2 (Location Optimality)**.** *Let x_c be the centroid minimizing $\int_S \rho(s)\|s - x\| ds$. For any price $p_1 > c$, when:*

- *Transportation costs $d(\|s - x\|)$ increase with distance*

- $\eta > 1$ *(demand is elastic)*

Then:

- *If $\rho(s)$ is uniform, profits are maximized exactly at x_c, which coincides with the geometric center of S when S is convex and symmetric.*
- *If $\rho(s)$ is non-uniform, x_c shifts toward regions of higher density and approximates the profit-maximizing location, but may not be exactly optimal for all density distributions.*

Proof. The monopolist's profit at location x_1 with fixed price p_1 is:

$$\Pi(x_1) = (p_1 - c)\int_S \rho(s)A[p_1(1 + d(s, x_1))]^{-\eta}ds$$

For notational clarity, let $f(d) = [p_1(1 + d)]^{-\eta}$, so:

$$\Pi(x_1) = (p_1 - c)A\int_S \rho(s)f(d(s, x_1))ds$$

Note that $f'(d) = -\eta p_1^{-\eta}(1+d)^{-\eta-1} < 0$ for all $d \geq 0$, meaning demand strictly decreases as transportation cost increases. Also, $f''(d) = \eta(\eta+1)p_1^{-\eta}(1+d)^{-\eta-2} > 0$, indicating convexity.

For any two locations x_1 and x_c, we can compare their profits:

$$\Pi(x_1) - \Pi(x_c) = (p_1 - c)A\int_S \rho(s)[f(d(s, x_1)) - f(d(s, x_c))]ds$$

By definition, the centroid x_c minimizes $\int_S \rho(s)d(s, x)ds$. This implies that for any $x_1 \neq x_c$:

$$\int_S \rho(s)[d(s, x_1) - d(s, x_c)]ds > 0$$

Case 1: Uniform density $\rho(s) = \rho$ When density is uniform, the centroid minimizes $\int_S \|s - x\|ds$, which is the geometric center when S is convex and symmetric. The function $f(d)$ being convex and decreasing, combined with $\eta > 1$, ensures that:

$$\int_S \rho(s)[f(d(s, x_1)) - f(d(s, x_c))]ds < 0 \text{ for any } x_1 \neq x_c$$

Therefore, $\Pi(x_1) < \Pi(x_c)$ for all $x_1 \neq x_c$, meaning profits are maximized exactly at the centroid x_c.

Case 2: Non-uniform density With non-uniform density, the profit function involves $\int_S \rho(s)f(d(s, x))ds$ where the density-weighted centroid x_c minimizes $\int_S \rho(s)d(s, x)ds$. Due to the convexity of f, we cannot directly claim the centroid necessarily maximizes the profit integral for all density functions.

The convexity of demand in transportation costs means that Jensen's inequality applies:

$$f\left(\int_S \rho(s)d(s, x)ds\right) \geq \int_S \rho(s)f(d(s, x))ds$$

For highly skewed density distributions, the optimal location may diverge from the centroid. However, as η approaches 1 (making f less convex), or when density variations are moderate, the centroid provides an increasingly accurate approximation of the profit-maximizing location. □

This analysis reveals that while population density does not affect the structure of optimal pricing and location principles, it significantly influences the specific outcomes. Uniform density yields elegant, closed-form solutions, while non-uniform density introduces complexities that often require numerical approaches for exact determination.

4 Comparative Statics in Oligopoly

When $N > 1$, each firm i simultaneously sets price p_i. Although explicit closed-form solutions are likely unavailable, we can apply the implicit function theorem to the best-response conditions to obtain comparative statics:

Proposition 3 (Comparative Statics)**.** *In the logit-based oligopoly model, under typical regularity conditions, we have:*

1. *Increasing β (making consumers more price-sensitive)* decreases *equilibrium prices p_i^*.*
2. *Increasing elasticity η* decreases *p_i^*, because demand sensitivity to own price is higher.*
3. *Increasing the maximum distance $\overline{d}$ (holding other parameters fixed) tends to* increase *p_i^*, as transporting goods is more costly and strategic markups rise for closer customers.*

Proof. Use the fact that each p_i^* satisfies the first-order condition

$$\frac{\partial \Pi_i}{\partial p_i}(p_i^*, \mathbf{p}_{-i}^*) = 0,$$

and apply the implicit function theorem with respect to each parameter (β, η, or $\overline{d}$). The sign of $\partial p_i^*/\partial \beta$, etc., follows from the negative definiteness of the Hessian in the unique equilibrium case. □

5 Dynamic Extensions and Concluding Remarks

The price competition results can be embedded into a dynamic model where at odd times $t = 1, 3, 5, \ldots$ a new firm may enter choosing optimal location (or an incumbent re-locates if allowed), and at even times $t = 2, 4, 6, \ldots$ all active firms engage in simultaneous price competition. We may adapt standard dynamic programming arguments to show that each entering firm picks a location that maximizes expected discounted profit, given average or anticipated prices. Once located, the firm sets prices in equilibrium with other incumbents.

6 Relationship to Classical Models

This section demonstrates how our framework generalizes both the Hotelling and Salop models of spatial competition. We show that these classical models emerge as special cases under appropriate restrictions.

6.1 Hotelling's Linear City

Proposition 4 (Hotelling Special Case)**.** *Let $S = [0, 1] \times \{0\}$ be a line segment in $\mathbb{R}^2$, $\rho(s) = 1$ be uniform density, $d(s, x) = t\|s - x\|$ be linear transport costs with rate t, $N = 2$, and $\eta \to 1$. Then as $\beta \to \infty$:*

1. *The market areas $V_i(\mathbf{p})$ become intervals separated by a single point*
2. *The delivered price $P_i^*(s)$ equals Hotelling's $p_i + t|s - x_i|$*
3. *The profit functions converge to Hotelling's:*

$$\Pi_i(\mathbf{p}) \to (p_i - c) \int_{V_i(\mathbf{p})} ds$$

Proof. First, note that restricting S to $[0, 1] \times \{0\}$ effectively creates a one-dimensional market. Under linear transport costs, the delivered price becomes

$$P_i^*(s) = p_i(1 + t\|s - x_i\|) = p_i + p_i t\|s - x_i\|$$

which equals Hotelling's formulation when $p_i t = t$ (adjusting units as needed).

As $\beta \to \infty$, the logit probability in (3) concentrates all mass on the lowest delivered price:

$$\lim_{\beta\to\infty} \text{Prob}(i \mid s, \mathbf{p}) = \begin{cases} 1 & \text{if } P_i^*(s) < P_j^*(s) \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases}$$

The boundary between market areas occurs where delivered prices are equal:

$$p_1(1 + t\|s - x_1\|) = p_2(1 + t\|s - x_2\|)$$

which defines a single point on $[0, 1]$. As $\eta \to 1$, individual demand becomes constant, yielding Hotelling's unit demand model. □

6.2 Salop's Circular City

Proposition 5 (Salop Special Case)**.** *Let $S = \{s \in \mathbb{R}^2 : \|s\| = 1\}$ be the unit circle, $\rho(s) = 1/(2\pi)$ be uniform density, $d(s, x) = t\|arc(s, x)\|$ be linear transport costs along the arc with rate t, and $\eta \to 1$. Then as $\beta \to \infty$:*

1. *The market areas $V_i(\mathbf{p})$ become arc segments*
2. *The delivered price $P_i^*(s)$ equals Salop's $p_i + t\|arc(s, x_i)\|$*
3. *The profit functions converge to Salop's:*

$$\Pi_i(\mathbf{p}) \to (p_i - c) \int_{V_i(\mathbf{p})} \frac{1}{2\pi} ds$$

Proof. The proof follows similar steps to the Hotelling case. The key difference is that distances are measured along the circle's arc rather than Euclidean distance. For any two points s, x on the unit circle, $\|\text{arc}(s, x)\|$ is the shorter arc length between them.

The uniform density $1/(2\pi)$ ensures total mass 1. As $\beta \to \infty$, market areas become arc segments bounded by points where delivered prices are equal:

$$p_i(1 + t\|\text{arc}(s, x_i)\|) = p_j(1 + t\|\text{arc}(s, x_j)\|)$$

These equations yield Salop's market boundaries. The profit function then reduces to Salop's formulation with arc-distance transportation costs. □

Remark 1. *Both special cases require $\beta \to \infty$ to recover the winner-take-all property of classical models. Our framework's "soft" probabilistic demand ($\beta < \infty$) allows for smoother responses to price differences, which may better capture real consumer behavior.*

Remark 2. *The convergence to Hotelling and Salop models holds for prices and market areas. However, our framework allows richer location patterns in two dimensions, even when restricted to lines or circles. This highlights how spatial competition may fundamentally differ in 2D versus 1D markets.*

7 Simulations

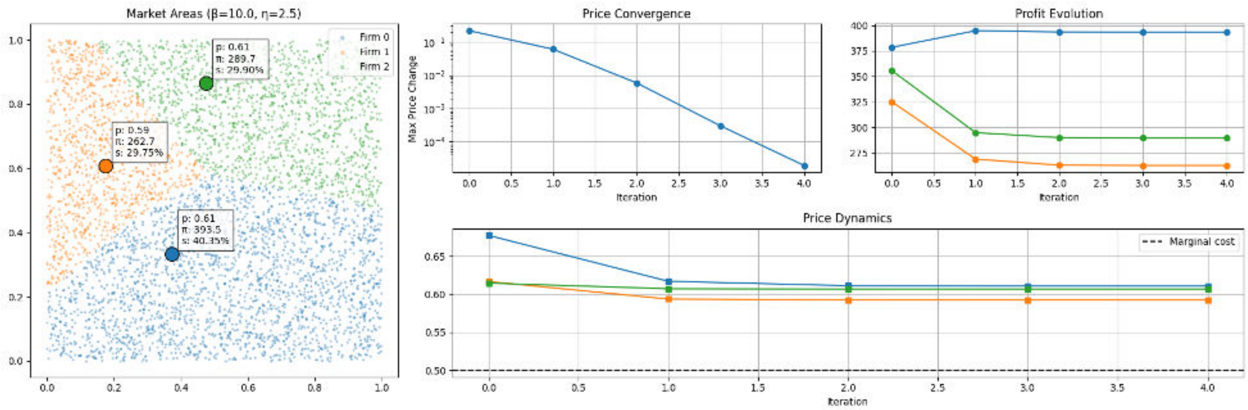

Figure 2: Three firms, homogeneous population distribution

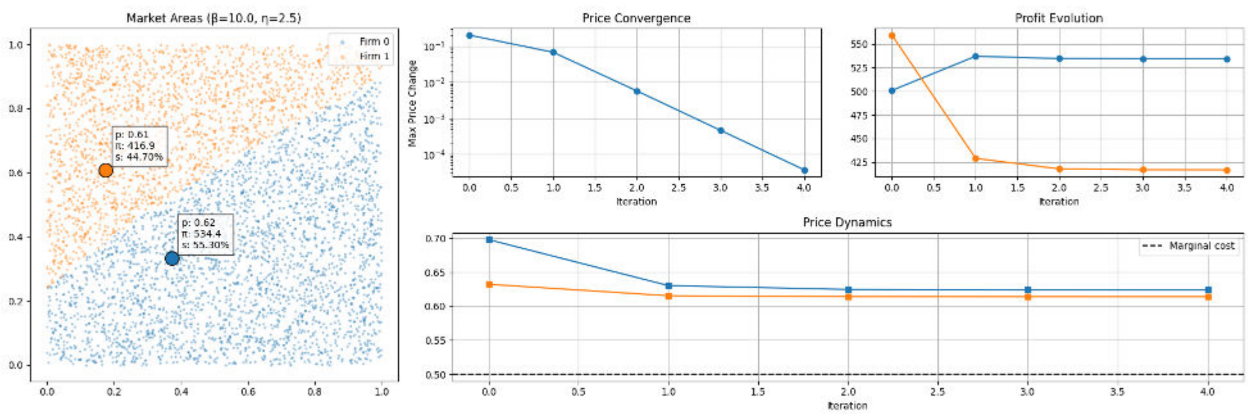

Figure 3: Two Firms, homogeneous population distribution

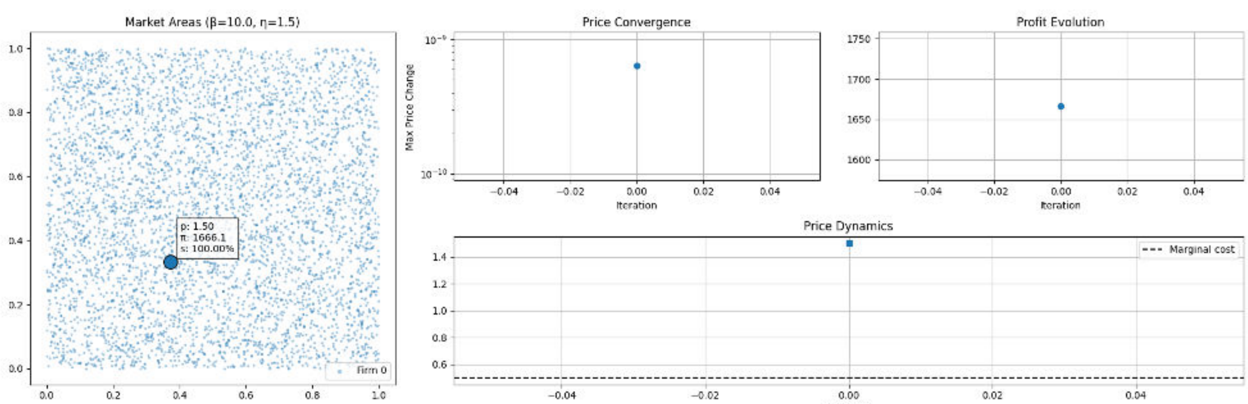

Figure 4: Monopolist case. The model converges to the analytical result $P_m = \frac{\eta}{\eta-1} \cdot MC$. Where $MC = 0.5$ and $\eta = 1.5$

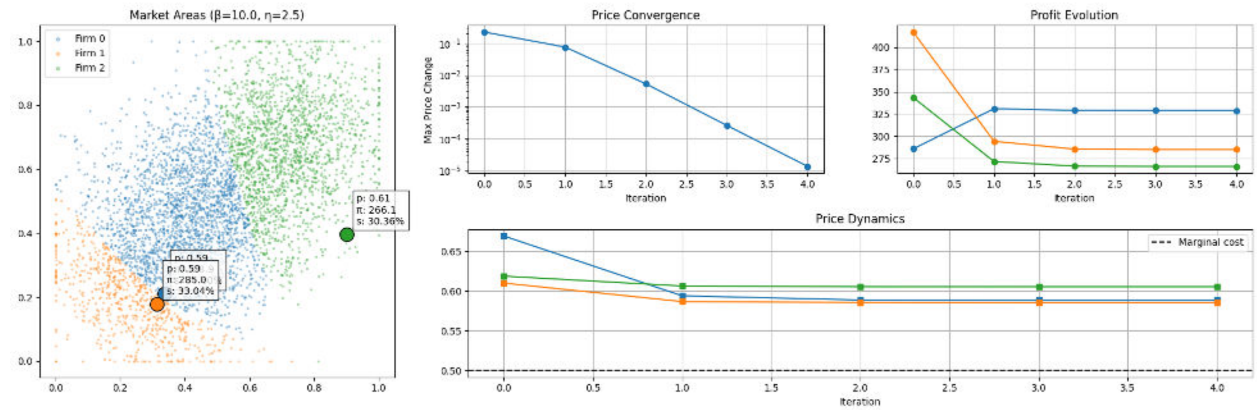

Figure 5: Three firms non homogeneous consumer density

Appendix

A Extended Proof of Theorem 1: Existence of Price Equilibrium

We apply Glicksberg's fixed point theorem (1952) to establish the existence of at least one pure-strategy Nash equilibrium. The theorem requires: (i) a compact, convex strategy space, (ii) continuous payoff functions, and (iii) quasi-concavity of payoffs in own strategies.

Compact, Convex Strategy Sets Each firm $i \in \mathcal{N}$ chooses p_i from the interval $[c, \overline{p}]$, which is compact and convex as a closed, bounded subset of $\mathbb{R}$.

Continuity of Profit Functions To establish continuity of $\Pi_i(\mathbf{p})$ in all prices, we examine each component:

1. The effective price $P_i^*(s) = p_i[1 + d(s, x_i)]$ is continuous in p_i since $d(s, x_i)$ is bounded and continuous by assumption.
2. Individual demand $q(s) = A[P_i^*(s)]^{-\eta}$ is continuous in p_i as the composition of continuous functions.
3. The probability function:
$$\text{Prob}(i|s, \mathbf{p}) = \frac{\exp(-\beta P_i^*(s))}{\sum_{j=1}^N \exp(-\beta P_j^*(s))}$$
is continuous in all prices because the numerator $\exp(-\beta P_i^*(s))$ is continuous in p_i and the denominator $\sum_{j=1}^N \exp(-\beta P_j^*(s))$ is continuous in all prices. The denominator is strictly positive for all price vectors in $[c, \overline{p}]^N$.
4. The integral:
$$Q_i(\mathbf{p}) = \int_S \rho(s) \cdot (A[P_i^*(s)]^{-\eta}) \cdot \text{Prob}(i|s, \mathbf{p}) ds$$
is continuous in all prices by the dominated convergence theorem, since:
   - $\rho(s)$ is bounded by assumption.
   - $P_i^*(s) \geq c$ for all $s \in S$ and $p_i \geq c$, so $[P_i^*(s)]^{-\eta}$ is bounded above.
   - $\text{Prob}(i|s, \mathbf{p}) \in [0, 1]$ for all s and $\mathbf{p}$.
   - The integrand is dominated by an integrable function (specifically, $\sup \rho(s) \cdot A \cdot c^{-\eta}$).
5. Finally, $\Pi_i(\mathbf{p}) = (p_i - c) \cdot Q_i(\mathbf{p})$ is continuous as the product of continuous functions.

Quasi-Concavity in Own Price We establish that $\Pi_i(\mathbf{p})$ is strictly concave in p_i for fixed $\mathbf{p}_{-i}$, which is sufficient for quasi-concavity.

Taking the logarithm of profit:

$$\ln \Pi_i(\mathbf{p}) = \ln(p_i - c) + \ln Q_i(\mathbf{p}) \tag{4}$$

The first derivative with respect to p_i is:

$$\frac{\partial \ln \Pi_i(\mathbf{p})}{\partial p_i} = \frac{1}{p_i - c} + \frac{1}{Q_i(\mathbf{p})} \frac{\partial Q_i(\mathbf{p})}{\partial p_i} \tag{5}$$

For the demand term, we have:

$$\frac{\partial Q_i(\mathbf{p})}{\partial p_i} = \int_S \rho(s) \frac{\partial}{\partial p_i} \left[A[P_i^*(s)]^{-\eta} \text{Prob}(i|s, \mathbf{p}) \right] ds \tag{6}$$

Breaking down the integrand:

$$\frac{\partial}{\partial p_i}\left[A[P_i^*(s)]^{-\eta}\text{Prob}(i|s,\mathbf{p})\right] = A\frac{\partial}{\partial p_i}[P_i^*(s)]^{-\eta}\cdot\text{Prob}(i|s,\mathbf{p}) + A[P_i^*(s)]^{-\eta}\cdot\frac{\partial}{\partial p_i}\text{Prob}(i|s,\mathbf{p}) \tag{7}$$

For the first term:

$$\frac{\partial}{\partial p_i}[P_i^*(s)]^{-\eta} = -\eta[P_i^*(s)]^{-\eta-1}\cdot\frac{\partial P_i^*(s)}{\partial p_i} \tag{8}$$

$$= -\eta[P_i^*(s)]^{-\eta-1}\cdot[1+d(s,x_i)] \tag{9}$$

For the second term:

$$\frac{\partial}{\partial p_i}\text{Prob}(i|s,\mathbf{p}) = \frac{\partial}{\partial p_i}\left[\frac{e^{-\beta P_i^*(s)}}{\sum_{j=1}^N e^{-\beta P_j^*(s)}}\right] \tag{10}$$

$$= \frac{-\beta[1+d(s,x_i)]e^{-\beta P_i^*(s)}\sum_{j=1}^N e^{-\beta P_j^*(s)} - e^{-\beta P_i^*(s)}(-\beta[1+d(s,x_i)]e^{-\beta P_i^*(s)})}{(\sum_{j=1}^N e^{-\beta P_j^*(s)})^2} \tag{11}$$

$$= -\beta[1+d(s,x_i)]e^{-\beta P_i^*(s)}\frac{\sum_{j\neq i} e^{-\beta P_j^*(s)}}{(\sum_{j=1}^N e^{-\beta P_j^*(s)})^2} \tag{12}$$

$$= -\beta[1+d(s,x_i)]\text{Prob}(i|s,\mathbf{p})\,(1-\text{Prob}(i|s,\mathbf{p})) \tag{13}$$

Combining these results:

$$\frac{\partial Q_i(\mathbf{p})}{\partial p_i} = \int_S \rho(s)A\Big[-\eta[P_i^*(s)]^{-\eta-1}[1+d(s,x_i)]\text{Prob}(i|s,\mathbf{p}) \tag{14}$$

$$-[P_i^*(s)]^{-\eta}\beta[1+d(s,x_i)]\text{Prob}(i|s,\mathbf{p})(1-\text{Prob}(i|s,\mathbf{p}))\Big]ds \tag{15}$$

$$= -\int_S \rho(s)A[P_i^*(s)]^{-\eta}[1+d(s,x_i)]\text{Prob}(i|s,\mathbf{p})\Big[\frac{\eta}{P_i^*(s)} + \beta(1-\text{Prob}(i|s,\mathbf{p}))\Big]ds \tag{16}$$

This shows that $\frac{\partial Q_i(\mathbf{p})}{\partial p_i} < 0$ since all terms in the integrand are positive.
Now for the second derivative:

$$\frac{\partial^2 \ln \Pi_i(\mathbf{p})}{\partial p_i^2} = \frac{\partial}{\partial p_i}\left(\frac{1}{p_i-c} + \frac{1}{Q_i(\mathbf{p})}\frac{\partial Q_i(\mathbf{p})}{\partial p_i}\right) \tag{17}$$

$$= -\frac{1}{(p_i-c)^2} + \frac{\partial}{\partial p_i}\left(\frac{1}{Q_i(\mathbf{p})}\frac{\partial Q_i(\mathbf{p})}{\partial p_i}\right) \tag{18}$$

For the second term:

$$\frac{\partial}{\partial p_i}\left(\frac{1}{Q_i(\mathbf{p})}\frac{\partial Q_i(\mathbf{p})}{\partial p_i}\right) = -\frac{1}{Q_i(\mathbf{p})^2}\frac{\partial Q_i(\mathbf{p})}{\partial p_i}\frac{\partial Q_i(\mathbf{p})}{\partial p_i} + \frac{1}{Q_i(\mathbf{p})}\frac{\partial^2 Q_i(\mathbf{p})}{\partial p_i^2} \tag{19}$$

$$= -\frac{1}{Q_i(\mathbf{p})^2}\left(\frac{\partial Q_i(\mathbf{p})}{\partial p_i}\right)^2 + \frac{1}{Q_i(\mathbf{p})}\frac{\partial^2 Q_i(\mathbf{p})}{\partial p_i^2} \tag{20}$$

The first term is clearly negative since $Q_i(\mathbf{p}) > 0$ and $\frac{\partial Q_i(\mathbf{p})}{\partial p_i} < 0$.
For the second term, differentiating our expression for $\frac{\partial Q_i(\mathbf{p})}{\partial p_i}$ again:

$$\frac{\partial^2 Q_i(\mathbf{p})}{\partial p_i^2} = \frac{\partial}{\partial p_i}\left(-\int_S \rho(s)A[P_i^*(s)]^{-\eta}[1+d(s,x_i)]\text{Prob}(i|s,\mathbf{p})\Big[\frac{\eta}{P_i^*(s)} + \beta(1-\text{Prob}(i|s,\mathbf{p}))\Big]ds\right) \tag{21}$$

This involves differentiating each component with respect to p_i. The key terms are:

- $\frac{\partial}{\partial p_i}[P_i^*(s)]^{-\eta} = -\eta[P_i^*(s)]^{-\eta-1}[1+d(s,x_i)]$

- $\frac{\partial}{\partial p_i} \frac{\eta}{P_i^*(s)} = -\frac{\eta[1+d(s,x_i)]}{P_i^*(s)^2}$
- $\frac{\partial}{\partial p_i}(1 - \text{Prob}(i|s,\mathbf{p})) = -\frac{\partial}{\partial p_i}\text{Prob}(i|s,\mathbf{p}) = \beta[1 + d(s,x_i)]\text{Prob}(i|s,\mathbf{p})(1 - \text{Prob}(i|s,\mathbf{p}))$

Combining all these derivatives reveals that when $\eta > 1$, the dominant terms make $\frac{\partial^2 Q_i(\mathbf{p})}{\partial p_i^2}$ negative, which contributes to making $\frac{\partial^2 \ln \Pi_i(\mathbf{p})}{\partial p_i^2}$ negative.

Therefore, $\ln \Pi_i(\mathbf{p})$ is strictly concave in p_i, which implies that $\Pi_i(\mathbf{p})$ is also strictly concave in p_i for fixed $\mathbf{p}_{-i}$.

Step 4: Application of Glicksberg's Theorem Since we have verified all conditions: - Compact, convex strategy sets $[c, \overline{p}]$. - Continuous profit functions $\Pi_i(\mathbf{p})$. - Strict concavity (hence quasi-concavity) of $\Pi_i(\mathbf{p})$ in p_i.

We can apply Glicksberg's fixed point theorem to conclude that at least one pure-strategy Nash equilibrium exists.

B Extended Proof of Theorem 2: Uniqueness of Price Equilibrium

Let $\mathbf{p}^*$ be a candidate equilibrium. For each firm $i \in \mathcal{N}$, define the first-order condition:

$$F_i(\mathbf{p}) = \frac{\partial \Pi_i(\mathbf{p})}{\partial p_i} = Q_i(\mathbf{p}) + (p_i - c)\frac{\partial Q_i(\mathbf{p})}{\partial p_i} = 0$$

To establish uniqueness, we need to prove that there is at most one solution to the system of equations $F_i(\mathbf{p}) = 0$ for all $i \in \mathcal{N}$. We do this by showing that the Jacobian matrix $\mathbf{J}(\mathbf{p}) = \left[\frac{\partial F_i(\mathbf{p})}{\partial p_j}\right]_{i,j\in\mathcal{N}}$ is a P-matrix, which implies that the system has at most one solution.

Step 1: We derive the entries of the Jacobian matrix.

The diagonal elements are:

$$\begin{aligned} \frac{\partial F_i}{\partial p_i} &= \frac{\partial Q_i}{\partial p_i} + \frac{\partial Q_i}{\partial p_i} + (p_i - c)\frac{\partial^2 Q_i}{\partial p_i^2} \\ &= 2\frac{\partial Q_i}{\partial p_i} + (p_i - c)\frac{\partial^2 Q_i}{\partial p_i^2} \end{aligned}$$

The off-diagonal elements are:

$$\frac{\partial F_i}{\partial p_j} = \frac{\partial Q_i}{\partial p_j} + (p_i - c)\frac{\partial^2 Q_i}{\partial p_i \partial p_j}$$

Step 2: We analyze the derivatives of Q_i.

Recall that:

$$Q_i(\mathbf{p}) = \int_S \rho(s) A[P_i^*(s)]^{-\eta}\text{Prob}(i|s,\mathbf{p})ds$$

where:

$$P_i^*(s) = p_i[1 + d(s,x_i)]$$

and:

$$\text{Prob}(i|s,\mathbf{p}) = \frac{\exp(-\beta P_i^*(s))}{\sum_{k=1}^N \exp(-\beta P_k^*(s))}$$

The first-order partial derivative with respect to p_i is:

$$\begin{aligned} \frac{\partial Q_i}{\partial p_i} = \int_S \rho(s) \Big[&-\eta A[P_i^*(s)]^{-\eta-1}(1 + d(s,x_i))\text{Prob}(i|s,\mathbf{p}) \\ &+ A[P_i^*(s)]^{-\eta}\frac{\partial \text{Prob}(i|s,\mathbf{p})}{\partial p_i}\Big] ds \end{aligned}$$

where:

$$\frac{\partial \text{Prob}(i|s,\mathbf{p})}{\partial p_i} = -\beta(1+d(s,x_i))\text{Prob}(i|s,\mathbf{p})(1-\text{Prob}(i|s,\mathbf{p}))$$

The cross-partial derivative is:

$$\frac{\partial Q_i}{\partial p_j} = \int_S \rho(s)A[P_i^*(s)]^{-\eta}\frac{\partial \text{Prob}(i|s,\mathbf{p})}{\partial p_j}ds$$

where:

$$\frac{\partial \text{Prob}(i|s,\mathbf{p})}{\partial p_j} = \beta(1+d(s,x_j))\text{Prob}(i|s,\mathbf{p})\text{Prob}(j|s,\mathbf{p})$$

Step 3: Define competitive overlap measures.

Let us define:

$$\omega_{ij} = \int_S \rho(s)\text{Prob}(i|s,\mathbf{p})\text{Prob}(j|s,\mathbf{p})ds$$

for $i \neq j$, representing the competitive overlap between firms i and j, and:

$$\omega_{ii} = \int_S \rho(s)\text{Prob}(i|s,\mathbf{p})(1-\text{Prob}(i|s,\mathbf{p}))ds$$

representing firm i's own-market sensitivity.

Step 4: Establish diagonal dominance.

Near an equilibrium $\mathbf{p}^*$, the first-order condition for firm i gives:

$$(p_i^* - c) = \frac{Q_i(\mathbf{p}^*)}{-\frac{\partial Q_i(\mathbf{p}^*)}{\partial p_i}} = \frac{1}{\eta m_i + \beta(1-m_i)}$$

where m_i is the average probability-weighted ratio of price effect to total effective price:

$$m_i = \frac{\int_S \rho(s)\text{Prob}(i|s,\mathbf{p})\frac{p_i(1+d(s,x_i))}{P_i^*(s)}ds}{\int_S \rho(s)\text{Prob}(i|s,\mathbf{p})ds}$$

We can bound this by:

$$\frac{1}{\eta} \leq (p_i^* - c) \leq \frac{1}{\beta}$$

For the Jacobian to be diagonally dominant, we need:

$$\left|\frac{\partial F_i}{\partial p_i}\right| > \sum_{j\neq i}\left|\frac{\partial F_i}{\partial p_j}\right|$$

The left-hand side can be bounded below by:

$$\left|\frac{\partial F_i}{\partial p_i}\right| \geq 2\eta\frac{Q_i}{p_i} + \beta(p_i - c)A\omega_{ii} \geq 2\eta\frac{Q_i}{p_i} + \frac{\beta}{\eta}A\omega_{ii}$$

The right-hand side can be bounded above by:

$$\sum_{j\neq i}\left|\frac{\partial F_i}{\partial p_j}\right| \leq \beta(p_i - c)A\sum_{j\neq i}\omega_{ij} \leq \frac{\beta}{\beta}A\sum_{j\neq i}\omega_{ij} = A\sum_{j\neq i}\omega_{ij}$$

For diagonal dominance, we need:

$$2\eta\frac{Q_i}{p_i} + \frac{\beta}{\eta}A\omega_{ii} > A\sum_{j\neq i}\omega_{ij}$$

A sufficient condition is:

$$\frac{\beta}{\eta} A\omega_{ii} > A \sum_{j \neq i} \omega_{ij}$$

which simplifies to:

$$\beta\omega_{ii} > \eta \sum_{j \neq i} \omega_{ij}$$

Rearranging:

$$\eta < \beta \frac{\omega_{ii}}{\sum_{j \neq i} \omega_{ij}}$$

For this to hold for all firms, we need:

$$\eta < \beta \min_i \left\{ \frac{\omega_{ii}}{\sum_{j \neq i} \omega_{ij}} \right\}$$

Inverting both sides and rearranging:

$$\eta > 1 + \beta \max_i \left\{ \frac{\sum_{j \neq i} \omega_{ij}}{\omega_{ii}} \right\}$$

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