Beyond the Beach: Spatial Competition in Two Dimensions

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Abstract

This paper extends Hotelling's spatial competition framework to two-dimensional markets with linear consumer demand and probabilistic firm choice. The model allows for general consumer density distributions across a compact geographic region, where firms compete in prices and locations and consumers choose according to a logit specification based on the lowest delivered price. We prove the existence of price equilibria and we extend the analysis to a dynamic framework with sequential firm entry and fixed entry costs, proving the existence of subgame perfect Nash equilibria. The model generates both minimum and maximum product differentiation depending on parameter values. Numerical simulations illustrate how the framework accommodates various market configurations and consumer distributions.

1. Introduction

The spatial dimension of firm behaviour was popularised by Hotelling (1929), who formalised a model in which firms simultaneously choose their location and pricing strategy to maximise profits. Hotelling —along with many subsequent researchers— assumed that both firms and consumers are located along a unit interval $[0, 1]$, representing a stylised "Main Street", where customers always choose the firm closest to them, and both firms set the same mill price for the good. As a result, competition occurs exclusively through location choices and competes to attract consumers based on distance and price, laying the foundation for an entire field of spatial competition theory. Remarkably, this idea had already appeared years earlier in the work of Launhardt (1885), whose contributions included a richer treatment of both horizontal and vertical differentiation. His model accounted for differing transportation technologies and strategic price setting as emphasised by Dos Santos Ferreira and Thisse (1996), who highlighted the pioneering value of Launhardt's approach.

Despite decades of research building on Hotelling's foundation, several fundamental limitations persist in the spatial competition literature. Most models remain constrained to one-dimensional

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’main street’ settings, limiting their applicability to real-world markets with complex geography. The standard assumption of uniform consumer distributions fails to capture the heterogeneous population patterns observed in actual markets, from urban density clusters to sparse rural areas. Perhaps most restrictively, the deterministic consumer choice assumption—where customers always patronize the nearest firm—overlooks the probabilistic nature of real consumer behaviour influenced by heterogeneous preferences, imperfect information, and random utility components. These simplifications, while analytically convenient, significantly limit our understanding of competition in realistic spatial environments.

This paper addresses these limitations by developing a generalized two-dimensional spatial competition framework that extends beyond Hotelling’s ’beach’. We allow for arbitrary consumer density distributions across compact geographic regions, incorporate probabilistic consumer choice through logit specifications, and analyze both static price competition and dynamic entry with fixed costs. In the static case, we prove the existence of price equilibria; in the dynamic framework, we establish the existence of subgame perfect Nash equilibria and provide analytical characterizations for the monopoly case. The model generates both minimum and maximum product differentiation depending on parameter values, with numerical simulations illustrating how the framework accommodates various market configurations. We also analyze the determinants of equilibrium market structure and provide comparative statics results.

Researchers have pursued several directions to address these limitations, though typically focusing on one aspect at a time. The most known modification is the circular model of Salop (1979) which addressed the boundary issues present in Hotelling’s line. More recent refinements include probabilistic consumer choice frameworks such as the conditional logit model introduced by McFadden (1974) and formalised for empirical use by Ben-Akiva and Lerman (1985). These models allow for smoother demand substitution across firms and constitute a more realistic setting for modern spatial pricing models.

Research on Hotelling’s work remains very active today. Scholars attempted to generalise the framework to various market configurations to gain additional insights when the market shape is not of the linear type. For example, Barro (2024), four decades after Salop’s initial work, reexamines a circular market, analyzing markups and firm entry. Similarly, Tsai and Lai (2005) proposed a triangular market structure as a step toward generalizing spatial market models. It is worth noting that in central place theory, optimal spatial organisation for a single good is also modelled as a hexagonal pattern around a central place, assuming uniform population density and equal accessibility. De Serpa (1986) however argued that the shape of the market (as long as it is a regular polygon) does not affect general equilibrium outcomes. As emphasized by Drezner and Drezner (2017) and Drezner and Eiselt (2024), when consumers are attracted to the nearest facility, each firm’s market share is proportional to the area where it is the closest provider. This geometric property underscores the importance of analysing the interaction between market shape, firm strategies, and consumer behaviour—a perspective that can open directions in spatial economic theory.

Another important aspect of spatial competition is the equilibrium location outcome of firms. In the economy, we observe both dispersion and agglomeration of firms. In his model, Hotelling introduced the concept of "minimum differentiation", which describes firms’ tendency to cluster around the centre of the market. In a spatial market, when the product is homogeneous, the only form of differentiation available to a firm is its location—hence the term minimum differentiation. However, the original Hotelling's formulation, as shown by D'Aspremont, Gabszewicz, and Thisse (1979), is problematic, and a Nash Equilibrium doesn't exist when firms are sufficiently near each other. When transport costs are linear, firms have an incentive to deviate slightly towards the centre to capture a larger market share, which makes any possible configuration unstable.

Although attempts to generalise Hotelling's original framework emerged as early as the 1970s, the issue remains an active field of research. As Cahan et al. (2021) points out, classic stylised two-dimensional models often lack realism and, for that reason, a network-based competition approach for a 2-D Hotelling game may be more suitable. Moreover, that work underlines how many attempts in the generalisation of Hotelling's model often describe the case in which consumers buy at the nearest location, while this is not an absolute rule. Relaxing this assumption Cox (1987) proposed a probabilistic choice model, where consumers are more likely to patronise nearby firms, but may still buy from more distant ones. This behaviour is captured by a probability vector over firm ranks by distance.[1] Larralde, Stehle, and Jensen (2009) proposed a spatial competition model in which consumer choice occurs according to a probabilistic logit rule, reflecting behaviour that is realistically influenced by heterogeneous preferences. Consumers are uniformly distributed over a continuous two-dimensional space, and each chooses between different firms based on the generalised price of the good, which includes the selling price and the transportation cost. The transport cost is quadratic with respect to the Euclidean distance between the consumer and the point of sale. Both the last works cited consider, however, more the location problem than the price.[2]

Probabilistic consumer behaviour has been studied by Wrede (2015) too, which has revisited the classic Hotelling model by incorporating probabilistic consumer preferences and endogenous residential choice. In this framework, firms simultaneously choose both location and price along a linear city, while consumers decide where to reside and from which firm to purchase, based on a logit model of utility that includes random preference heterogeneity. Introducing a continuous logit structure guarantees the existence and uniqueness of a subgame-perfect Nash equilibrium,—a result that resolves the classical non-existence problem highlighted by D'Aspremont, Gabszewicz, and Thisse (1979) under linear demand. This contribution, however, is "constricted" to 1-D modelling. Not only that, but the majority of the models consider, as in Hotelling's original work, a duopoly on a main street. Despite the maintenance of this last aspect, Brenner (2005) extends the classic Hotelling model to markets with three or more firms competing in location and price, assuming quadratic transportation costs. He shows that, unlike the duopoly case, equilibria with more players exhibit neither maximal nor minimal differentiation. Firms tend to adopt asymmetric and interior positions, and pricing follows a U-shaped pattern, with corner firms often pricing more aggressively. While explicit solutions are derived for up to five players, Brenner uses numerical simulations for up to nine firms, demonstrating the existence and structure of equilibria for a general number of players.

[1] The classical Hotelling model corresponds to the "degenerate" case where the entire probability mass is on the nearest firm

[2] In Larralde, Stehle, and Jensen (2009), in particular, the price is the same for all enterprises, set at a common value, so competition is exclusively spatial, also if is included in the calculation of consumer utility but not subject to strategic optimization

A further common feature of the literature discussed above is the assumption of a uniform distribution of consumers across one- or two-dimensional spaces. While this assumption has been widely explored (Neven 1986), it remains a relevant point of discussion, as more general spatial competition models—allowing for non-uniform distributions—can encompass several classical frameworks as special cases. For instance, the Bertrand duopoly can be seen as a specific instance of Hotelling competition with a non-uniform consumer distribution, as in Rashid (2016). In his model, the price-setting stage mirrors Hotelling's when consumers are clustered near the two firms located at a given distance. Here, the distance determines the price gap required to induce switching: if prices are in this range, firms share the market; otherwise, the cheaper firm captures all demand. These perspectives point to important directions for future research, suggesting that it is possible and necessary to move toward more general models of spatial competition that better reflect the heterogeneity of real-world markets, both in terms of geometry and consumer behavior. From this point of view, even the Hotelling original setup itself could be seen as a special case of a more comprehensive theory.

The paper proceeds as follows: Section 2 presents the core static model and establishes existence results for price equilibria. Section 3 extends this framework to incorporate dynamic entry with fixed costs, including a detailed analysis of the monopoly case as an initial entry scenario and the properties of the resulting market structure. Section 4 explores the model's behaviour through numerical simulations, illustrating various market outcomes and analysing phenomena such as spatial differentiation. Section 5 concludes. Proofs and technical details appear in the Appendices.

2. The static Model

Consider a compact, convex region $S \subset \mathbb{R}^2$ describing a geographic market. Consumers are located in the space and are immobile. Let $\rho : S \to \mathbb{R}_+$ denote a continuous, bounded consumer density with total mass

$$M = \int_S \rho(s)\, ds.$$

We assume $M > 0$ is finite.

In the market, there is a number N of firms selling a homogeneous good, indexed by $\mathcal{N} = \{1, \dots, N\}$, each located at $x_i \in S$. Firms compete by setting prices $p_i \in [c, \overline{p}]$, where $c > 0$ is the common marginal cost. The upper bound $\overline{p}$ ensures positive demand throughout the strategy space: $\overline{p} < \min_{i,s}(\alpha/\gamma - t \cdot d(s, x_i))$, assuming this range is non-empty and c is sufficiently low.

Consumer Behavior and Demand

The effective price paid by a consumer at location s buying from firm i is:

$$P_i^*(s) = p_i + t \cdot d(s, x_i), \tag{1}$$

where $t > 0$ is the transportation cost rate and $d(s, x_i) : S \times S \to [0, \overline{d}]$ represents the transportation cost function, assumed continuous in both arguments and convex in x_i.

Consumers have linear demand. The quantity demanded by a consumer at location s from

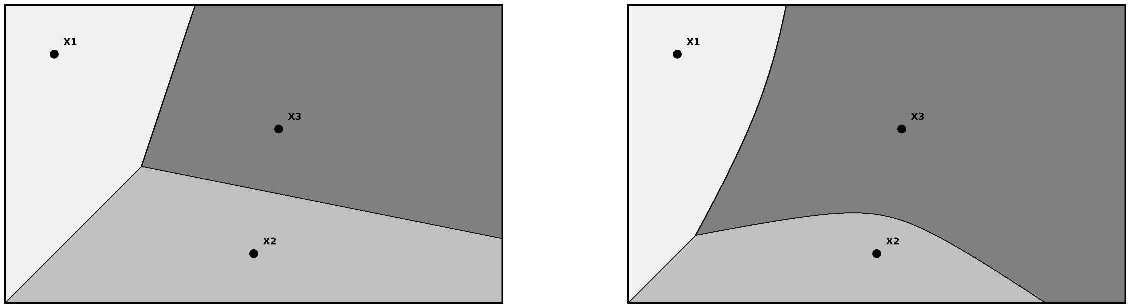


FIGURE 1. Market areas defined by lowest effective price (similar to weighted Voronoi). The left panel shows the case when all firms have the same price. The right panel shows the case when one firm lowers its price.

firm i is:

$$q_i(s) = \alpha - \gamma P_i^*(s), \tag{2}$$

where $\alpha > 0$ and $\gamma > 0$ are demand parameters.

Consumers buy the homogeneous good from the firm that sells it at the lowest effective price that is the price augmented with the transport cost. We model the consumer choice with a logit approach similar to Larralde, Stehle, and Jensen (2009); thus, the probability that a consumer at location s chooses firm i is:

$$\text{Prob}\big(i \mid s, \mathbf{p}\big) = \frac{\exp\big(-\beta\, P_i^*(s)\big)}{\sum_{j=1}^{N} \exp\big(-\beta\, P_j^*(s)\big)}, \tag{3}$$

where $\beta > 0$ is the logit parameter reflecting consumer sensitivity to effective price differences. It is worth noting that as $\beta \to \infty$, the model approaches the deterministic "lowest effective price" rule as in Hotelling.

Figure 1 illustrates how market areas change when firm prices vary, showing the case of equal prices (left panel) and the effect when one firm lowers its price (right panel).

The spatial demand for firm i, given the price vector $\mathbf{p} = (p_1, \ldots, p_N)$, is:

$$Q_i(\mathbf{p}) = \int_S \rho(s)\,(\alpha - \gamma P_i^*(s))\,\text{Prob}\big(i \mid s, \mathbf{p}\big)\, ds. \tag{4}$$

The Optimal Price Problem

In the static scenario, where a fixed number of immobile firms already exists in the economy, each firm optimizes its profits by adjusting prices depending on the locations of the firms and the prices set by competitors. The function that each firm seeks to maximize is:

$$\Pi_i(\mathbf{p}) = (p_i - c)\, Q_i(\mathbf{p}) \tag{5}$$

$$\max_{p_i \in [c, \overline{p}]} \Pi_i(\mathbf{p}) \tag{6}$$

We now establish the existence of price equilibria in this spatial competition framework.

THEOREM 1 (Existence of Price Equilibrium). *Under the assumptions above (continuous* ρ*,* $\alpha > 0, \gamma > 0$*,* $\beta > 0$*,* $t > 0$*, finite* $\overline{p} > c$*, compact strategy sets* $[c, \overline{p}]$ *where* $\overline{p}$ *ensures* $q_i(s) > 0$*), and a logit consumer choice function with linear individual demand, there exists at least one pure-strategy price equilibrium.*

PROOF. See Appendix A. □

The existence proof applies Glicksberg's fixed point theorem, using the properties of continuous payoffs, compact convex strategy sets, and quasi-concavity of profit functions.

While the uniqueness of price equilibria can be established under certain conditions (when the direct own-price sensitivity of demand is sufficiently strong relative to competitive cross-price effects), we focus here on existence, which is sufficient for our analysis of the dynamic entry game.

3. Dynamic Entry with Fixed Costs

We now extend our model to incorporate firm entry decisions with fixed costs, creating a dynamic framework that endogenizes market structure. This allows us to analyze how the first entrant and subsequent entrants shape the market.

The Dynamic Entry-Location-Price Game

Consider an infinite-horizon dynamic game with entry cost $F > 0$ and a common discount factor $\delta \in (0, 1)$:

- **Odd Periods** ($t = 1, 3, \dots$) **(Entry Stage):** A potential entrant (firm $N_{t-1}+1$) observes the locations $\mathbf{x}_{t-1}$ of incumbent firms and decides whether to enter. If entry occurs, the entrant chooses location x_E^* to maximize anticipated profit:

$$\max_{x_E \in S} \Pi_E^*(\mathbf{x}_{t-1}, x_E) - F \tag{7}$$

  where $\Pi_E^*(\mathbf{x}_{t-1}, x_E)$ is the anticipated profit in the subsequent price competition stage. Let $V_E(\mathbf{x}_{t-1}) = \max_{x_E \in S} \Pi_E^*(\mathbf{x}_{t-1}, x_E)$ represent the maximum attainable profit for a potential entrant. Entry occurs if and only if $V_E(\mathbf{x}_{t-1}) > F(1 - \delta)$.
- **Even Periods** ($t = 2, 4, \dots$) **(Price Competition Stage):** Given locations $\mathbf{x}_t$, firms simultaneously set prices, leading to the equilibrium price vector $\mathbf{p}^*(\mathbf{x}_t)$ as characterized in Section 2.

Initial Entry: The Monopoly Case

The first firm to enter the market, for the first two periods, faces no competition and acts as a monopolist. This special case offers analytical solutions for both optimal pricing and location.

PROPOSITION 1 (Monopoly Price Formula). *For a monopolist at location* x_1*, the profit-maximizing price is:*

$$p_1^*(x_1) = \frac{\alpha}{2\gamma} + \frac{c}{2} - \frac{t \cdot \bar{d}_\rho(x_1)}{2}, \tag{8}$$

where $\bar{d}_\rho(x_1) = \frac{1}{M}\int_S \rho(s)d(s,x_1)ds$ *is the weighted average distance to consumers. This solution assumes* $\alpha - \gamma c - \gamma t\bar{d}_\rho(x_1^*) \geq 0$ *to ensure* $p_1^*(x_1^*) \geq c$, $p_1^*(x_1^*) \leq \overline{p}$ *to guarantee positive demand quantities, and that all consumers in the market consume from the monopolist.*

PROOF. For a monopolist at x_1, profit is $\Pi(p_1) = (p_1 - c)\int_S \rho(s)(\alpha - \gamma[p_1 + t \cdot d(s,x_1)])ds$. Differentiating with respect to p_1 and setting to zero:

$$\begin{aligned}\int_S \rho(s)(\alpha - \gamma[p_1 + td(s,x_1)])ds + (p_1 - c)\int_S \rho(s)(-\gamma)ds &= 0\\ M(\alpha - \gamma p_1 - \gamma t\bar{d}_\rho(x_1)) - \gamma M(p_1 - c) &= 0\\ \alpha - \gamma p_1 - \gamma t\bar{d}_\rho(x_1) - \gamma p_1 + \gamma c &= 0\\ \alpha + \gamma c - \gamma t\bar{d}_\rho(x_1) &= 2\gamma p_1\end{aligned}$$

Therefore:

$$p_1^* = \frac{\alpha}{2\gamma} + \frac{c}{2} - \frac{t\bar{d}_\rho(x_1)}{2} \tag{9}$$

The second derivative $\partial^2\Pi/\partial p_1^2 = -2\gamma M < 0$ confirms this is a maximum. □

PROPOSITION 2 (Monopolist Location). *The monopolist chooses the location* x_1^* *that minimizes the weighted average distance to consumers* $\bar{d}_\rho(x_1)$.

PROOF. Substituting the optimal price into the profit function:

$$\Pi(x_1) = \frac{M}{4\gamma}(\alpha - \gamma c - \gamma t\bar{d}_\rho(x_1))^2 \tag{10}$$

Assuming $\alpha - \gamma c - \gamma t\bar{d}_\rho(x_1) > 0$ (ensuring positive demand), maximizing profit requires minimizing $\bar{d}_\rho(x_1)$. □

REMARK 1. *For uniform consumer density ($\rho(s)$ = constant) in a convex region S and when using Euclidean distance, the optimal monopolist location is the geometric median of S. For general convex distance functions, the solution corresponds to the* L^1*-medoid.*

REMARK 2. *The optimal price decreases with average distance, capturing how the monopolist must lower prices to serve distant consumers due to their higher transportation costs.*

The first firm will enter the market if and only if the discounted monopoly profit exceeds the entry cost:

$$\frac{\Pi(x_1^*)}{1 - \delta} > F \tag{11}$$

This condition determines whether the market can sustain at least one firm. If satisfied, the first entrant locates at x_1^* and sets price $p_1^*(x_1^*)$ as characterized above.

Equilibrium Analysis with Multiple Firms

After the first entry, subsequent firms may enter sequentially, each choosing their optimal location given the existing configuration of incumbents.

THEOREM 2 (Existence of SPNE with Entry). *Under the assumptions of our model, with a positive fixed entry cost $F > 0$, a common discount factor $\delta \in (0, 1)$, and assuming the price equilibrium exists for any configuration of firms, there exists a subgame perfect Nash equilibrium (SPNE) in the infinite-horizon entry-location-price game.*

PROOF. See Appendix C. □

THEOREM 3 (Non-Uniqueness of SPNE with Entry). *The entry-location-price game generally admits multiple subgame perfect Nash equilibria, distinguished by different numbers of firms, different location configurations, or both.*

PROOF. See Appendix C. □

These theorems establish that an equilibrium market structure will emerge, but typically not a unique one. The multiplicity arises from various sources: multiple optimal locations for each entrant, path dependence in sequential entry decisions, strategic entry deterrence by incumbents, and the integer constraint on the number of firms.

Equilibrium Properties

PROPOSITION 3 (Market Structure Properties). *The equilibrium number of firms N^* in the entry-location-price game with fixed cost F:*

a. *Decreases with F (higher entry costs lead to fewer firms)*
b. *Increases with market size M (larger markets support more firms)*
c. *Decreases with parameters that intensify price competition (β and γ)*
d. *Increases with parameters that soften price competition (larger t)*

PROOF. See Appendix C. □

The characterization of equilibrium market structure provides insights into how market fundamentals shape industry concentration. Higher entry costs and more intense price competition lead to more concentrated markets, while larger market size and greater spatial differentiation support more firms in equilibrium.

PROPOSITION 4 (Comparative Statics). *Assuming a price equilibrium exists for any configuration of firms, we obtain the following comparative statics:*

a. *Increasing β (more price-sensitive consumer choice) generally leads to* lower *equilibrium prices p_i^*.*
b. *Increasing γ (higher price sensitivity of demand) generally leads to* lower *equilibrium prices p_i^*.*
c. *Increasing t (higher transportation costs) tends to* increase *equilibrium prices p_i^* by softening competition through greater spatial differentiation.*
d. *Increasing α (higher base demand) generally leads to* higher *equilibrium prices p_i^*.*

PROOF. See Appendix B. □

These comparative statics align with economic intuition. Higher values of β and γ intensify price competition by making consumers more responsive to price differences, driving prices downward. Conversely, higher transportation costs (t) increase spatial differentiation, softening price competition and allowing firms to charge higher prices. Higher base demand (α) increases consumers' willingness to pay, enabling firms to charge higher prices.

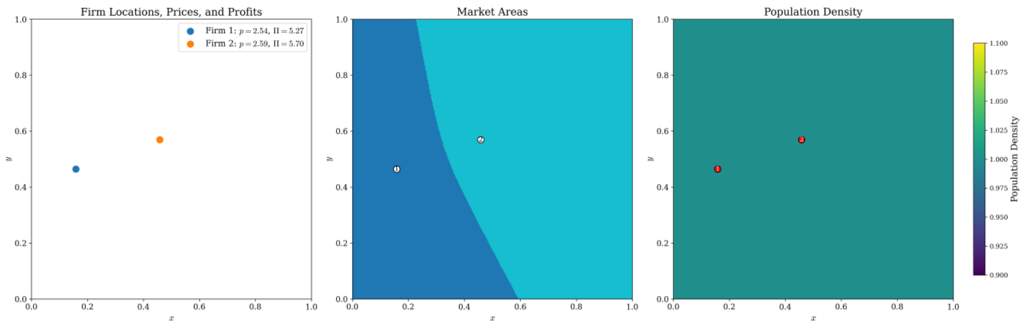


FIGURE 2. Two Firms, homogeneous population distribution.

4. Equilibrium Analysis and Simulations

While the model guarantees the existence of a price equilibrium, its specific characteristics for $N > 1$ are analytically intractable. We therefore turn to numerical simulation to explore the model's properties and show the market outcomes under different conditions. The simulations are conducted in a square market $S = [0, 1]^2$ with Euclidean distance.

We first illustrate the baseline static competitive equilibrium presented in section 2. Figure 2 depicts a duopoly where two randomly located firms compete in the market with a homogeneous consumer distribution ($\rho(s) = 1$). Each firm generates its own market area based on their locations and their prices. The probabilistic nature of consumer choice implies that while the solid lines represent the loci where consumers are indifferent based on effective price, firms still compete for consumers on the "wrong" side of the line, albeit with a lower probability. The consumers are more rational with higher β values, meaning that they have strong preferences even when the difference among the effective prices are small.

Consider now the dynamic model. At times $t = 1, 2$ there is only one firm in the economy which will choose location first and then its price. The first firm is unaware whether new firms will enter the market and will behave in the first two periods as a monopolist. We derived in proposition 1 that in our framework the monopolist price depends linearly on the trade costs. As the transport cost t approaches zero, the market becomes non-spatial, and the optimal price should converge to the monopoly price with a linear demand, $p_m = \frac{\alpha}{2\gamma} + \frac{c}{2}$. Figure 3 shows the simulated profit-maximizing price for a monopolist as t varies. The numerical result converges exactly to the analytical benchmark.

After the first and second periods, new firms can choose to enter the market until entry is no longer profitable.

If the assumptions are further relaxed, allowing for non-homogeneous population density, we obtain interesting results. Figure 4 shows an equilibrium with five firms in a market where consumers are distributed according to a 3 foci Gaussian distribution. The first firm that enters the market, as for proposition 2 maximizes its profits by choosing the position that minimizes the weighted average distance to consumers. However, the new entering firms choose nearer locations to each Gaussian focus, progressively reducing the monopolist profits. When the simulation ends, the location of the first firm that entered the market is no longer the one generating more profits.

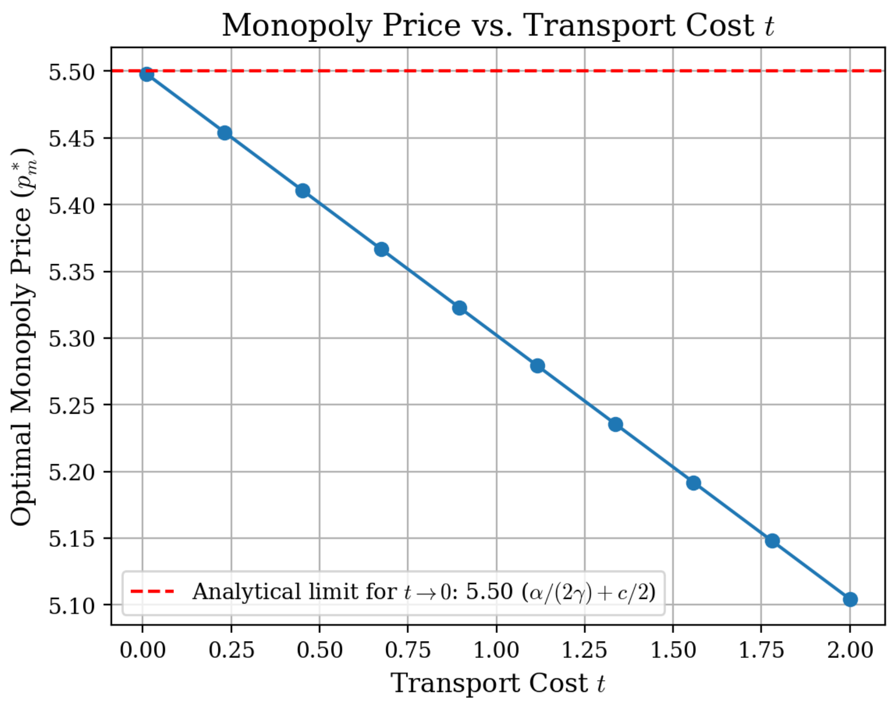


FIGURE 3. Monopolist case. The model with linear demand $q = \alpha - \gamma P^*$ and negligible transport costs ($t \to 0$) converges to the analytical result $p_m = \frac{\alpha}{2\gamma} + \frac{c}{2}$.

4.1. Spatial Differentiation: Agglomeration vs. Dispersion

Beyond location relative to consumers, the model endogenously determines the location of firms relative to each other. This speaks to the classic question of minimum versus maximum differentiation (D'Aspremont, Gabszewicz, and Thisse 1979; Palma et al. 1985). The model reveals a key trade-off: a "demand pull" encourages firms to locate near density peaks, while a "competitive push" encourages them to locate far from rivals to soften price competition. The balance is governed by market parameters, chiefly the consumer choice sensitivity β.

PROPOSITION 5 (Spatial Differentiation Regimes). *In an equilibrium:*

a. ***Agglomeration Regime:*** *For sufficiently low values of β, firms tend to cluster.*
b. ***Dispersion Regime:*** *For sufficiently high values of β and transport cost t, firms tend to disperse to*

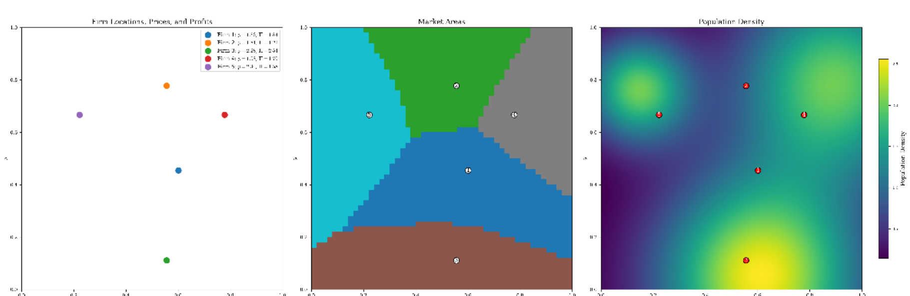
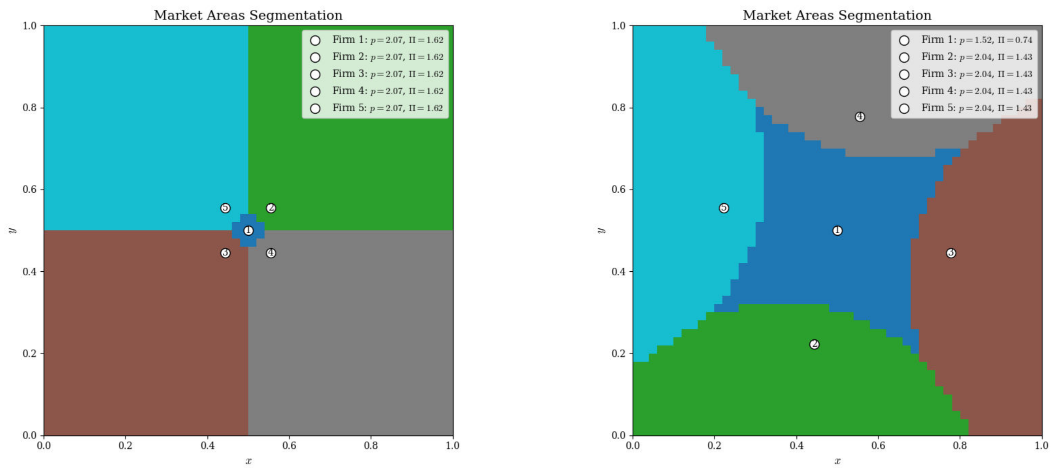


FIGURE 4. Three firms with non-homogeneous consumer density.



(a) Agglomeration Regime (Low β) (b) Dispersion Regime (High β, high t)

FIGURE 5. Equilibrium locations for five firms under different differentiation regimes in a market with a central density peak. The background shading indicates consumer density.

soften price competition.

Figure 5 illustrates this proposition computationally. In a market with homogeneous population distribution, two firms enter sequentially. With low β, all the firms cluster near the center to access the largest pool of consumers (Panel a). With high β and t, the incentive to avoid head-to-head price competition dominates, pushing the firms into a dispersed configuration (Panel b).

5. Conclusion

This paper develops a two-dimensional spatial competition model that extends the traditional Hotelling framework in several directions. We incorporate linear consumer demand, probabilistic choice through a logit specification, and allow for general consumer density distributions across a two-dimensional market space. Our main theoretical contributions are threefold. First, we establish the existence of price equilibria for the static game using Glicksberg's fixed point theorem, demonstrating that the model produces well-defined competitive outcomes. Second, we extend the framework to incorporate sequential firm entry with fixed costs, showing that subgame perfect Nash equilibria exist in the dynamic setting, though typically multiple equilibria arise due to location multiplicity and path dependence. Third, we provide analytical characterizations for the monopoly case, showing that the optimal location minimizes weighted average distance to consumers while the optimal price decreases linearly with this average distance. The model demonstrates how spatial differentiation emerges endogenously through the interaction of demand concentration and competitive forces. Our simulations illustrate that firms may either agglomerate near demand centers or disperse to avoid price competition, depending on the strength of consumer price sensitivity and transportation costs. The framework captures both minimum and maximum differentiation as special cases of a more general trade-off. While the model nests classical one-dimensional frameworks like Hotelling and Salop as limiting cases, the two-dimensional setting with probabilistic choice provides additional flexibility for analyzing realistic market configurations. The combination of analytical tractability for key cases (monopoly, existence results) with numerical methods for exploring complex multi-firm scenarios offers a practical approach for studying spatial competition. The framework opens several natural directions for extension, including dynamic firm relocation, quality differentiation, and welfare analysis. The model's structure makes it suitable for empirical applications where both consumer location data and firm pricing strategies are observable.

A. Extended Proof of Theorem 1: Existence of Price Equilibrium

We apply Glicksberg's fixed point theorem (1952) to establish existence of a pure-strategy Nash equilibrium.

Compact, Convex Strategy Sets. Each firm $i \in \mathcal{N}$ chooses $p_i \in [c, \overline{p}]$. This interval is compact and convex. The assumption on $\overline{p}$ is that $\overline{p} < \min_{i,s}(\alpha/\gamma - t \cdot d(s, x_i))$ and $c < \overline{p}$, ensuring $q_i(s) = \alpha - \gamma P_i^*(s) > 0$ for all $s \in S$ and $p_i \in [c, \overline{p}]$.

Continuity of Profit Functions. $\Pi_i(\mathbf{p}) = (p_i - c)Q_i(\mathbf{p})$. The firm's i demand is $Q_i(\mathbf{p}) = \int_S \rho(s)\,(\alpha - \gamma P_i^*(s))\,\text{Prob}\big(i \mid s, \mathbf{p}\big)\,ds$. Notice that:

a. $P_j^*(s) = p_j + t \cdot d(s, x_j)$ is continuous in p_j.
b. $q_i(s) = \alpha - \gamma P_i^*(s)$ is continuous in p_i (and positive by assumption on $\overline{p}$).
c. $\text{Prob}(i|s, \mathbf{p}) = \exp(-\beta P_i^*(s))/\sum_k \exp(-\beta P_k^*(s))$ is continuous in all p_k. Denominator is strictly positive.
d. The integrand $\rho(s)q_i(s)\text{Prob}(i|s, \mathbf{p})$ is continuous in $\mathbf{p}$ and bounded on $S \times [c, \overline{p}]^N$ (since ρ bounded, $q_i(s) \in (0, \alpha]$, Prob $\in [0, 1]$).
e. By dominated convergence, $Q_i(\mathbf{p})$ is continuous in $\mathbf{p}$.
f. $\Pi_i(\mathbf{p})$ is continuous as a product of continuous functions.

Quasi-Concavity in Own Price. We show that $\ln \Pi_i(\mathbf{p})$ is concave in p_i for $p_i > c$ (so $\Pi_i > 0$ as $Q_i > 0$) where $\ln \Pi_i(\mathbf{p}) = \ln(p_i - c) + \ln Q_i(\mathbf{p})$.

Notice that $\ln(p_i - c)$ is strictly concave in p_i so we just need $\ln Q_i(\mathbf{p})$ to be concave in p_i too.

Remember that $Q_i(\mathbf{p}) = \int_S K_i(s, p_i, \mathbf{p}_{-i})ds$, where $K_i(s, p_i, \mathbf{p}_{-i}) = \rho(s)(\alpha - \gamma P_i^*(s))\text{Prob}(i|s, p_i, \mathbf{p}_{-i})$.

Since $\alpha - \gamma P_i^*(s) > 0$ for all $p_i \in [c, \overline{p}]$ (by our assumption on $\overline{p}$), its logarithm $\ln(\alpha - \gamma(p_i + td(s, x_i)))$ exists and is concave in p_i.

$\ln \text{Prob}(i|s, p_i, \mathbf{p}_{-i})$ is concave in p_i (second derivative $-\beta^2\text{Prob}(i)(1 - \text{Prob}(i)) \leq 0$).

Since $\ln \rho(s)$ is constant w.r.t p_i, $\ln K_i(s, p_i, \mathbf{p}_{-i})$ is a sum of concave functions, hence concave in p_i. So K_i is log-concave.

By Prékopa-Leindler theorem, $Q_i(\mathbf{p})$ is log-concave in p_i. Thus, $\ln Q_i(\mathbf{p})$ is concave in p_i.

Therefore, $\ln \Pi_i(\mathbf{p})$ is concave in p_i. This implies $\Pi_i(\mathbf{p})$ is quasi-concave in p_i.

Application of Glicksberg's Theorem. All conditions are met, so at least one pure-strategy Nash equilibrium exists.

B. Comparative Statics Derivations

In this appendix, we derive the comparative statics results presented in Proposition 4.

B.1. Preliminaries

Let $\mathbf{p}^*$ denote the equilibrium price vector. At equilibrium, each firm's first-order condition must be satisfied:

$$F_i(\mathbf{p}^*) = \frac{\partial \Pi_i(\mathbf{p}^*)}{\partial p_i} = Q_i(\mathbf{p}^*) + (p_i^* - c)\frac{\partial Q_i(\mathbf{p}^*)}{\partial p_i} = 0 \quad \forall i \in \mathcal{N} \tag{12}$$

To analyze how equilibrium prices change with respect to a parameter $\theta \in \{\beta, \gamma, t, \alpha\}$, we apply the implicit function theorem to the system of equations $F_i(\mathbf{p}^*, \theta) = 0$. Define the Jacobian matrix $J = [\frac{\partial F_i}{\partial p_j}]_{i,j}$. The changes in equilibrium prices with respect to θ are given by:

$$\frac{d\mathbf{p}^*}{d\theta} = -J^{-1} \cdot \frac{\partial \mathbf{F}}{\partial \theta} \tag{13}$$

where $\frac{\partial \mathbf{F}}{\partial \theta} = [\frac{\partial F_i}{\partial \theta}]_i$ is the vector of partial derivatives with respect to θ.

Under conditions where the Jacobian J is invertible, and if all elements of $\frac{\partial \mathbf{F}}{\partial \theta}$ have the same sign, and the off-diagonal elements of $-J^{-1}$ are non-negative, the corresponding elements of $\frac{d\mathbf{p}^*}{d\theta}$ will have the same sign as $-\frac{\partial F_i}{\partial \theta}$. This allows us to determine the direction of price changes.

B.2. Effects of Changing β (Choice Intensity)

For the choice intensity parameter β, we have:

$$\frac{\partial F_i}{\partial \beta} = \frac{\partial Q_i}{\partial \beta} + (p_i^* - c)\frac{\partial^2 Q_i}{\partial p_i \partial \beta} \tag{14}$$

As β increases, consumers become more price-sensitive, making $\frac{\partial Q_i}{\partial p_i}$ more negative. This increased elasticity pushes firms to reduce prices.

At equilibrium, this effect typically dominates, leading to:

$$\frac{\partial F_i}{\partial \beta} < 0 \tag{15}$$

When this holds for all firms, we have:

$$\frac{dp_i^*}{d\beta} < 0 \tag{16}$$

B.3. Effects of Changing γ (Price Sensitivity)

For the demand price sensitivity parameter γ, we have:

$$\frac{\partial F_i}{\partial \gamma} = \frac{\partial Q_i}{\partial \gamma} + (p_i^* - c)\frac{\partial^2 Q_i}{\partial p_i \partial \gamma} \tag{17}$$

As γ increases, each consumer's quantity demanded decreases for a given effective price. Additionally, higher γ makes demand more responsive to price changes.

Both effects work in the same direction, resulting in:

$$\frac{\partial F_i}{\partial \gamma} < 0 \tag{18}$$

When this holds for all firms, we have:

$$\frac{dp_i^*}{d\gamma} < 0 \tag{19}$$

B.4. Effects of Changing t (Transportation Cost)

For the transportation cost parameter t, as t increases, spatial differentiation becomes more important relative to price differences, making demand less price-elastic. This softening of price competition is the dominant effect, yielding:

$$\frac{\partial F_i}{\partial t} > 0 \tag{20}$$

When this holds for all firms, we have:

$$\frac{dp_i^*}{dt} > 0 \tag{21}$$

B.5. Effects of Changing α (Base Demand)

For the base demand parameter α, higher base demand increases quantity demanded proportionally across all firms, resulting in:

$$\frac{\partial F_i}{\partial \alpha} > 0 \tag{22}$$

When this holds for all firms, we have:

$$\frac{dp_i^*}{d\alpha} > 0 \tag{23}$$

C. Analysis of Dynamic Entry Game

This appendix provides a detailed analysis of the dynamic entry game with fixed costs presented in Section 3. We prove the existence of subgame perfect Nash equilibria (SPNE) and explain why multiple equilibria typically arise.

C.1. Proof of Theorem 2: Existence of SPNE

We prove existence of an SPNE through three key steps:

Step 1: Bounded Number of Firms

We first establish that the entry process eventually terminates at some finite number of firms.

LEMMA 1 (Bounded Entry). *There exists a finite upper bound $\bar{N}$ such that if $N_t \geq \bar{N}$, no further entry will occur in any SPNE.*

PROOF. The total market profit is bounded above. Let Π^{max} denote the maximum industry profit achievable:

$$\Pi^{max} \leq \int_S \rho(s)(\alpha - \gamma c)ds \times (\overline{p} - c) \tag{24}$$

This bound applies because:

- The maximum quantity sold is bounded by $\int_S \rho(s)(\alpha - \gamma c)ds$, which occurs when effective prices are at their minimum $P_i^*(s) = c$.
- The maximum markup is bounded by $(\overline{p} - c)$.

As the number of firms N increases, the maximum profit any individual firm can earn decreases. With $c > 0$ ensuring equilibrium prices bounded away from zero, we can show that there exists some $\bar{N}$ such that for any configuration of $\bar{N}$ firms and any location choice by a potential $(\bar{N}+1)$-th entrant:

$$\Pi^*_{\bar{N}+1}(\mathbf{x}_{\bar{N}}, x_{\bar{N}+1}) < F(1-\delta) \tag{25}$$

This means the discounted profit stream from entry would be less than the entry cost, making entry unprofitable. Thus, entry stops at or before $\bar{N}$ firms. □

Step 2: Existence of Optimal Strategies at Each Stage

LEMMA 2 (Price Equilibrium Existence). *For any number of firms N and location configuration $\mathbf{x}_N$, a pure-strategy Nash equilibrium of the price-setting game exists.*

PROOF. This follows directly from Theorem 1. The price-setting game satisfies the conditions of Glicksberg's theorem. □

LEMMA 3 (Optimal Location Choice). *For any configuration of incumbent firms $\mathbf{x}_{inc}$, if entry is profitable, there exists an optimal location choice for the entrant.*

PROOF. The entrant's profit function $\Pi^*_E(\mathbf{x}_{inc}, x_E) = (p^*_E - c)Q_E(\mathbf{p}^*)$ is continuous in x_E when the price equilibrium depends continuously on locations, which is the case when the equilibrium exists (though it may not be unique). Since S is compact, by the Weierstrass theorem, $\Pi^*_E(\mathbf{x}_{inc}, \cdot)$ attains its maximum on S. Let $x^*_E \in \arg\max_{x_E \in S} \Pi^*_E(\mathbf{x}_{inc}, x_E)$. If $\Pi^*_E(\mathbf{x}_{inc}, x^*_E) > F(1-\delta)$, entry at x^*_E is optimal; otherwise, not entering is optimal. □

Step 3: Construction of SPNE

With the previous lemmas established, we can construct an SPNE for the infinite-horizon game:

a. Start with an empty market ($N_0 = 0$).
b. For each period t, define strategies as follows:
    - Odd periods (entry stage): If $N_{t-1} < \bar{N}$, compute $V_E(\mathbf{x}_{t-1}) = \max_{x_E \in S} \Pi^*_E(\mathbf{x}_{t-1}, x_E)$. If $V_E(\mathbf{x}_{t-1}) > F(1-\delta)$, enter at $x^*_E \in \arg\max_{x_E \in S} \Pi^*_E(\mathbf{x}_{t-1}, x_E)$; otherwise, do not enter.
    - Even periods (price competition): Play the Nash equilibrium of the price-setting game, yielding $\mathbf{p}^*(\mathbf{x}_t)$.

This strategy profile forms an SPNE by construction:

- In price-setting stages, firms play a Nash equilibrium by definition.
- In entry stages, potential entrants make optimal entry and location decisions given the continuation play.
- The discount factor $\delta \in (0, 1)$ ensures present-value payoffs are well-defined.
- Since the number of firms is bounded by $\bar{N}$, the entry process terminates in finite time, making the specified strategy profile well-defined for the entire game.

C.2. Proof of Theorem 3: Multiplicity of Equilibria

We now explain why multiple SPNE typically exist. The proof is constructive, highlighting sources of multiplicity.

Source 1: Multiple Optimal Locations

LEMMA 4 (Location Multiplicity). *There exist market configurations where an entrant has multiple optimal locations that yield equivalent profits.*

PROOF. Consider a symmetric case with $\rho(s)$ uniform on a disc S, and two incumbents located at opposite sides of the disc's circumference. By symmetry, a third entrant has at least two optimal locations that yield identical profits. Each of these location choices leads to a different market configuration and affects the entry decisions and optimal locations for subsequent entrants. □

Source 2: Path Dependence

LEMMA 5 (Path Dependence). *Different sequences of entry decisions can lead to different equilibrium market structures.*

PROOF. Let $\mathbf{x}_A = (x_1^A, x_2^A, \ldots, x_{N_A}^A)$ and $\mathbf{x}_B = (x_1^B, x_2^B, \ldots, x_{N_B}^B)$ be two different location configurations reached through different entry sequences. Because of the differences in locations, $V_E(\mathbf{x}_A)$ and $V_E(\mathbf{x}_B)$ may differ. It's possible that $V_E(\mathbf{x}_A) > F(1-\delta)$ while $V_E(\mathbf{x}_B) < F(1-\delta)$, meaning entry continues in path A but stops in path B. This leads to equilibria with different numbers of firms. □

These sources of multiplicity establish that the entry-location-price game typically admits multiple SPNE, completing the proof of Theorem 3.

C.3. Market Structure Properties

We now provide the reasoning for the comparative statics on market structure presented in Proposition 3.

PROPOSITION 6 (Entry Cost Effect). *The equilibrium number of firms N^* is non-increasing in the entry cost F.*

PROOF. The entry condition is $V_E(\mathbf{x}_{N-1}) > F(1-\delta)$. Increasing F makes this inequality harder to satisfy, which cannot increase the number of firms that enter in equilibrium. □

PROPOSITION 7 (Market Size Effect). *The equilibrium number of firms N^* is non-decreasing in the market size $M = \int_S \rho(s)ds$.*

PROOF. A proportional increase in $\rho(s)$ across all locations increases each firm's profit proportionally. This makes the entry condition easier to satisfy for a given number of firms, potentially supporting more entrants in equilibrium. □

PROPOSITION 8 (Competition Parameters). *The equilibrium number of firms N^* is:*

a. *Non-increasing in β (choice intensity parameter)*
b. *Non-increasing in γ (demand price sensitivity)*
c. *Non-decreasing in t (transportation cost parameter)*

PROOF. From Section B, we established that:

- Higher β and γ intensify price competition, reducing equilibrium prices and profits
- Higher t softens price competition, increasing equilibrium prices and profits

Lower profits make the entry condition $V_E(\mathbf{x}_{N-1}) > F(1-\delta)$ harder to satisfy, while higher profits make it easier to satisfy. The effects on N^* follow accordingly. □

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